EVALUATION OF A CLASS OF SEQUENTIAL SAMPLING PROCEDURES

by

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THESIS

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ABSTRACT

A class of sequential procedures for estimating the mean of a normal distribution having known variance from quantal response data is discussed. This class includes as special members the up-and-down method and other procedures commonly used in biological assay. A method of evaluating alternative procedures belonging to a given subset of the class is presented. This method is essentially an application of Wald's decision theory. A loss plus cost objective function is used and the efficiency of a particular procedure is determined by its ability to satisfy one of the four criteria considered. Criteria are discussed for use with both the expected value and variance of the total loss, which may be determined from matrix equations that are derived. Two applications are given. The first is an application to procedures commonly used in biological assay. In the second, an application to the elevation procedure of the precision registration technique used by U.S. Army and Marine Corps field artillery units, it is seen that under certain conditions, Dixon's modified up-and-down method strictly dominates the elevation procedure currently in use.

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I. INTRODUCTION

The up-and-down method is an experimental procedure for estimating the median of a continuous distribution function G from quantal response data. The procedure, which originated from testing the sensitivity of explosives to shock, is applicable to experimental situations which deal with continuous variables which cannot be measured [6].

The biological assay problem, for example, is an application which has been studied by Dixon and Mood [6], Brownlee, Hodges, and Rosenblatt [2], Dixon [4], Tsutakawa [15] and others. The objective in the bio-assay problem is to determine the critical dose of a drug above which greater than fifty per cent of the test specimens respond. As Dixon and Mood visualized the problem,

> "... a critical dose is associated with each insect (test specimen), but one cannot measure it. He can only try some dose and observe whether or not the critical dose for that insect is less than or greater than the chosen dose." [6].

In performing a bio-assay experiment a prescribed number of specimens is usually tested at each of several fixed dose levels. If the up-and-down method, originally studied by Dixon and Mood, is used, some initial dose level in a set of equally spaced dose levels is chosen for testing the first specimen. The next specimen is tested at the dose level immediately below or above the dose level of the previous test depending upon whether there was or was not a response

on the previous test, and so forth [5]. Note the structure of this procedure. A trial is performed using one specimen. After the initial trial, the level used in succeeding trials of the experiment is determined by the response from the preceeding trial. Tsutakawa [15] studied a generalization of the up-and-down method in which the number of specimens tested per trial is a constant which is at least one. In the first trial of Tsutakawa's procedure, which he calls a "random walk design", a group of $k \geq 1$ observations (that is, $k \geq 1$ specimens are tested) is made at some level Y_1 in a set of equally spaced levels. Here, each observation in a trial is a response or nonresponse depending upon whether the value of the random variable having distribution G is less than or greater than some known value; that is, if the specimen responds, then it is inferred that the critical value (the median of G) is less than the value used to test the specimen. Succeeding trials of the experiment are made at levels $Y_2, \ldots, Y_n^*$, with k observations per trial, determined by:

$$Y_{t+1} = \begin{cases} Y_t + m & \text{if } 0 \leq J_t \leq k^o \\ Y_t & \text{if } k^o < J_t < k - k^o, \; t \geq 1 \\ Y_t - m & \text{if } k - k^o \leq J_t \leq k \end{cases} \qquad (1)$$

where m is the interval between successive levels, J_t is the number of responses on the t-th trial, and k^o is a predetermined integer such that $0 \leq k^o < k - k^o$. This

procedure generates the sequence $\{Y_t, J_t\}$ and is designated $W(k, k^o)$ by Tsutakawa. Note the differences between the up-and-down method and $W(k, k^o)$. In the latter, a trial is performed using k specimens and the level used in succeeding trials of the experiment is determined by the number of responses from the preceeding trial. Notice also that the design W(1, 0) is identically the up-and-down method.

The procedures of concern in this study are sequential sampling procedures for estimating the mean of G. The purpose of this study is twofold: to extend Tsutakawa's procedures, and to formulate a methology for evaluating alternative procedures. The procedures to be studied differ from Tsutakawa's procedures, $W(k, k^o)$ in that:

1) It is assumed that G is a normal distribution having known variance.

2) It is assumed that $\mathcal{M}$, a set of equally spaced levels, is finite.

3) The number N* of trials taken in the experiment is a random variable.

4) The number k of observations taken per trial is not necessarily constant throughout the experiment.

5) Successive trials of the experiment are not necessarily performed at successive levels in $\mathcal{M}$.[1]

---

[1] In discussing his results, Tsutakawa [15] suggests that the experimental situation may prompt a widening of his class of procedures wherein k and m in equation 1 are permitted to vary. The properties 4) and 5) above permit such variations.

Thus, if $\mathcal{M}$ is a finite set of equally spaced levels, then $\mathcal{C}$, the class of procedures to be studied, is defined as a class of procedures for estimating the mean of a normal distribution, having known variance, from quantal response data which have the properties 3), 4) and 5) given above. When both 4) and 5) are restricted (that is, k is constant and successive trials are performed at successive levels), then Tsutakawa's procedures, $W(k, k^{\circ})$, belong to $\mathcal{C}$, a case which is discussed separately as a special case of the general procedure in $\mathcal{C}$.

It is assumed that the experimenter has selected a nominal sample size or a nominal number of trials for the experiment and can select a set of equally spaced levels (this set is a subset of $\mathcal{M}$) for the experiment which include the mean of G. Here, the term "nominal" is used in the same sense as used by Dixon [4]; that is, as first suggested by Brownlee, et al [2], some of the observations, usually a string of responses or nonresponses, are discarded as being indicative of a poorly selected initial level. To perform an experiment using the general procedure, the experimenter selects a stopping rule which depends upon the sequence of responses. He chooses the interval between trials; that is, whether successive trials are to be performed at successive levels or at irregular levels. For example, he may desire that the spacing between trials decrease when the responses obtained indicate he is testing near the mean. He then decides upon the number of

observations per trial to be taken and the observation/response combinations which determine whether the next trial will be performed at a higher level, a lower level, or the same level. For instance, in the up-and-down method the observation/response combinations are 1/0 and 1/1. In the first case, the next trial would be performed at a higher level; while in the second case, it would be performed at a lower level. These choices can be made in advance because the experimenter knows the nominal sample size. After experimentation, the estimate of the mean is determined using one of several estimators.

The general procedure is evaluated by representing the estimate of the mean, derived by use of the procedure, as the level nearest the estimate. This is an approximate representation which can be made as accurate as one pleases by increasing the number of levels while decreasing the spacing between levels. Thus, the total loss, in deciding the estimate is at a particular level, can be taken as the loss associated with deciding that it is at some level plus the cost of sampling. Hence, the problem of determining an optimal procedure W* in Γ, a set of procedures in $\mathcal{W}$ under consideration, is the decision problem formulated by Wald [17]. Matrix equation, derived for the expected value and variance of the total loss, are used with one of four criteria discussed to determine the optimal procedure.[2] The

[2]This approach was suggested by the approach used in a report by Elfving [14].

criteria considered consist of the usual Bayes and minimax criteria used in conjunction with the expression for the expected loss. In addition, two analogously defined criteria are used in conjunction with the expression for the variance of the loss. In the special case, it is shown that choosing the number k^*, of observations per trial which minimizes the expected value of the total loss is equivalent to finding a procedure which satisfies the Bayes criterion.

The present study was motivated by a gunnery problem encountered by U.S. Army and Marine Corps field artillery units. The problem is that of determining a basis for corrections to an element of firing data called elevation. Here, elevation is an angle to which a gun tube must be raised in order that the center of impact of a group of projectiles fired from the tube will be at a given distance, called the range, from the gun. The procedure, currently used to determine the adjusted elevation, is known as the elevation procedure of the field artillery registration technique. While the origin of the current procedure is obscure, it is known to have evolved from earlier procedures in use since World War II. After an adjustment phase in which the center of impact is moved near a known point referred to as the registration point, the procedure essentially consists of: observing whether a projectile bursts beyond or short of the registration point; then, changing the elevation setting after a prescribed sequence of

responses have been observed. This pattern of experimentation is continued until a nominal sample size of six bursts have been observed. Then, the adjusted elevation is computed.

Notice the similarity in the problem of determining the adjusted elevation and the bio-assay problem. Paraphrasing Dixon and Mood, an adjusted elevation is associated with each registration point, but one cannot measure it. He can only try some elevation setting and observe whether the projectile bursts beyond or short of the registration point, that is, observe whether the adjusted elevation is less than or greater than the chosen elevation setting.

It was discovered that Tsutakawa's procedures, which include an earlier version of the elevation procedure, do not include the procedure currently in use because of properties 4) and 5) listed above for procedures in the class $\mathcal{C}$. Thus in order to evaluate the current procedure and some alternative procedures, it was necessary to consider a wider class of procedures.

The reader interested primarily in the application to the elevation procedure of the precision registration technique may omit Sections IIB and IIIC. The reader interested in the more general aspects of the problem may omit Section IIIC-2 and Chapter IV.

II. THE GENERAL PROCEDURE

An extension of Tsutakawa's procedures will now be made precise. Let $\mathcal{M}' = \{\ldots, m_{-1}, m_0, m_1, \ldots\}$ be a set of equally spaced levels with interval m where the m_i are midpoints of the set of intervals

$$\mathcal{J} = \left\{\ldots, \left(\frac{-3m}{2}, \frac{-m}{2}\right], \left(\frac{-m}{2}, \frac{m}{2}\right], \left(\frac{m}{2}, \frac{3m}{2}\right], \ldots\right\}. \tag{2}$$

For later convenience, let m_0 be the mean of G. Let $\mathcal{M}''$, a finite subset of $\mathcal{M}'$ containing m_0, be known to the experimenter; that is, assume that the experimenter has sufficient knowledge of the random variable having distribution G to select a subset of the levels of $\mathcal{M}'$ for the experiment which contains m_0. This can be done without serious loss of generality because the variance of G is known and the experimenter need not know the specific level in $\mathcal{M}''$ which is m_0. It is further assumed that the experiment begins at some randomly selected level in $\mathcal{M}''$ but, in general, some of the trials of the experiment may be performed at levels not in $\mathcal{M}''$. Thus, in the experiment described below, a random number N* of trials is performed at the levels in $\mathcal{M}'$.

In the general procedure, the first trial of the experiment is performed at level Y_1 in $\mathcal{M}''$ where a set of k_1 observations are taken. On the $n^* - 1$ succeeding trials of the experiment, $k_2, \ldots, k_{n^*}$ observations are taken at levels

$Y_2, \ldots, Y_n^*$ determined by:

$$Y_{t+1} = \begin{cases} Y_t + C_t \cdot m & \text{if } 0 \leq J_t \leq k_t^{o} \\ Y_t & \text{if } k_t^{o} < J_t < k_t - k_t^{o} \\ Y_t - C_t \cdot m & \text{if } k_t - k_t^{o} \leq J_t \leq k_t \end{cases}, \quad t \geq 1 \tag{3}$$

where $\{C_t: C_t - 1 \leq C_{t+1} \leq C_t + 1\}$[3] is a sequence of predetermined integers which determine the spacing between levels at which trials are performed; $\{k_t\}$ is a sequence of predetermined integers which determines the number of observations per trial; $\{k_t^{o}\}$ is a sequence of predetermined integers such that $0 \leq k_t^{o} < k_t - k_t^{o}$; and J_t is the number of responses on the t-th trial. Thus, the experimenter selects a level in $\mathcal{Y}$ for the first trial of the experiment. Succeeding trials are performed at levels determined by use of the sequences $\{C_t\}$, $\{k_t\}$, and $\{k_t^{o}\}$ which are known to the experimenter and J_t which the experimenter observes. Each observation is a response or nonresponse depending upon whether the value of the random variable having the normal distribution G is less than or greater than some known value.

The number N* of trials of the experiment is determined by use of a stopping rule δ. Here, the set of stopping rules under consideration is such that N* is finite and a particular rule δ depends upon the sequence $\{J_t\}$ in a manner similar to that suggested by Brownlee, et al [2] and

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[3]The restriction on $\{C_t\}$ is discussed below.

used by Dixon [4] in his study of the up-and-down method for small sample sizes. For example, if $k_t = 1$ for all t, then a typical rule δ might be to perform six trials of the experiment after the first reversal (that is, first response after a string of nonresponses, or vice versa) in the sequence $\{J_t\}$. Thus, for a given performance of the experiment, the number N of observations is determined by the rule δ and the sequence $\{k_t\}$ from

$$N = \sum_{t=1}^{N^*} k_t \, . \tag{4}$$

The general procedure described above generates the sequence $\{Y_t\}$, which forms a random walk on the levels in $\mathcal{M}'$, and determines N. The general procedure is a modified Tsutakawa random walk procedure and will be designated $W(\delta, k_t, k_t^o, C_t)$ or simply W when no confusion results from the abbreviation. Notice the differences between Tsutakawa's $W(k, k^o)$ and $W(\delta, k_t, k_t^o, C_t)$. Instead of a fixed number of observations, the latter uses a stopping rule to determine the number of observations. Sequences rather than constants are used in $W(\delta, k_t, k_t^o, C_t)$ to determine the level of each trial, the spacing between levels used for successive trials and the number of observations per trial. Notice also that the procedure $W(\delta, k_t, k_t^o, C_t) = W(\delta, k, k^o, 1)$ is identical to Tsutakawa's procedure $W(k, k^o)$ except for the rule δ, which has been incorporated into the former. Procedures of the type

$W(\delta, k, k^{o}, 1)$ will be referred to as a special case of the general procedure in [illegible] and will be discussed separately below.

In the above, the general procedure has been described with the performance of an experiment in mind. That is, the information known to the experimenter has been given and the procedure he should follow has been outlined. In that which follows, the discussion will begin to focus on the second purpose of the study--to formulate a methodology for evaluating alternative procedures in [illegible]. Here, it is convenient to view the general procedure from the standpoint of an experimenter interested in determining which procedure among a set of alternative procedures in the class is best.[4] When viewed in this light, the general procedure can be characterized by two stochastic matrices which facilitate the evaluation.

A. TWO MATRICES WHICH CHARACTERIZE THE GENERAL PROCEDURE

If a single observation is made of the random variable having distribution G, then α_i and β_i, the probabilities of response and nonresponse at the i^{th} level in [illegible], are

$$\alpha_i = G(m_i) = 1 - \beta_i; \; i = 0, \pm 1, \ldots \quad . \tag{5}$$

Note that if $i = 0$, then $\alpha_0 = G(m_0) = 0.5 = \beta_0$. Also, it is clear from the symmentry of G that there exists levels in

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[4] The sense in which a procedure is best or optimal will be clarified below. The term efficiency is also used to mean the degree of optimality of a procedure.

$\mathcal{M}'$, equidistant from m_0, say m_{-u} and m_u, such that

$$\alpha_{-u} = G(m_{-u}) \doteq 0, \qquad (6)$$

$$\alpha_u = G(m_u) \doteq 1,$$

so that levels m_{-u} and m_u may be taken as reflecting barriers. Let $\mathcal{M}$ be such a subset of $\mathcal{M}'$; that is, let $\mathcal{M} = \{m_{-u}, \ldots, m_0, \ldots, m_u\}$ where the index u is chosen such that $\mathcal{M}''$, the set of levels known to the experimenter, is contained in $\mathcal{M}$. Thus, it follows that the experiment begins at some randomly selected level in $\mathcal{M}$. Note however that if $C_t \neq 1$ for some t, then the $(t+1)^{th}$ trial of the experiment may be performed at some level not in $\mathcal{M}$. Suppose, for example that the t-th trial is performed at level m_{u-1} and that $C_t = \nu$, $\nu \neq 1$. Then it is possible that the $(t+1)^{th}$ trial will be performed at level $m_{u+\nu-1}$, a level which is outside the reflecting barrier at level m_u. To get around this difficulty,[5] all the levels outside m_{-u} and m_u may also be taken as reflecting barriers and the restriction, $C_t - 1 \leq C_{t+1} \leq C_t + 1$, imposed on the sequence $\{C_t\}$. Under these assumptions, it follows that the sequence $\{Y_t\}$ cannot contain two successive levels not in $\mathcal{M}$.

When the experiment ends, the experimenter may compute an estimate of the mean of G, say $\hat{m}_0$, using one of several

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[5]Note that this difficulty does not exist in the special case where $C_t = 1$ for all t.

estimators which will be generically denoted $\hat{Y}$. This computation may be performed using all or part of the data in the sequence $\{Y_t, J_t\}$ together with N, N*, and m. Now $\hat{m}_0$ is in one of the intervals of $\mathcal{J}$, given by equation 2, the midpoint of which is given by one of the levels in $\mathcal{M}'$; that is, $\hat{m}_0$ may be represented by one of the levels in $\mathcal{M}'$. And since the sequence $\{Y_t\}$ cannot contain two successive levels not in $\mathcal{M}$, it is reasonable to assume in the general procedure that $\hat{m}_0$ may be represented by one of levels in $\mathcal{M}$. This assumption is not necessary in the special case because there Y_t is in $\mathcal{M}$ for all t. Thus, when it is said that the experiment ends at a level in $\mathcal{M}$, it is meant that $\hat{m}_0$ is in $\mathcal{M}$.

The sequence $\{Y_t\}$ generated by W can be classified as an integer valued discrete time stochastic process having state space $\mathcal{M}' = \{\ldots, -1, 0, 1, \ldots\}$ and parameter space $T' = \{1, 2, \ldots\}$ where in $\mathcal{M}'$, $m_0 = 0$ and the interval $m = 1$. Expressions for the one-step transition probabilities that, on the $(t+1)^{th}$ trial of the experiment, the process goes up a level,[6] remains at the same level, or goes down a level may be found using (3) and the fact that J_t is distributed binomially with parameters k_t and α_i, which is given by (5).

---

[6]To perserve the terminology of the up-and-down method the term "level" will be used in lieu of the term "state."

Let

$$p_{i,j} = \begin{cases} p_i & j = i + C_t \\ r_i & j = i \\ q_i & j = i - C_t \\ 0 & \text{otherwise} \end{cases} \qquad (i = 0, \pm 1, \ldots) \qquad (7)$$

denote the probability that the process goes to the j-th level on the (t+1) trial given it was at the i-th level on the t-th trial. Then it follows from (3) that

$$\begin{aligned} p_i &= \Pr(0 \le J_t \le k_t^{\,o}) = \sum_{j=0}^{k_t^{\,o}} \binom{k_t}{j} \alpha_i^j \beta_i^{k_t - j} \\ r_i &= \Pr(k_t^{\,o} < J_t < k_t - k_t^{\,o}) = \sum_{j=k_t^{\,o}+1}^{k_t - k_t^{\,o} - 1} \binom{k_t}{j} \alpha_i^j \beta_i^{k_t - j} \\ q_i &= \Pr(k_t - k_t^{\,o} \le J_t \le k_t) = \sum_{j=k_t - k_t^{\,o}}^{k_t} \binom{k_t}{j} \alpha_i^j \beta_i^{k_t - j} \end{aligned} \qquad (8)$$

where $p_i \ge 0$, $r_i \ge 0$, $q_i \ge 0$ and $p_i + r_i + q_i = 1$; $i = 0, \pm 1, \ldots$.

Now consider the triplet $(\mathcal{M}, W, \hat{Y})$. Let $\hat{m}_0$, determined by use of $\hat{Y}$, be represented by one of the levels in $\mathcal{M}$. Recall that when the experiment is performed, sampling begins at some level in $\mathcal{M}$. Now the actual level in $\mathcal{M}$ at which sampling begins is a random variable, say Q, which depends upon prior knowledge of the experimental situation. Likewise the level in $\mathcal{M}$ which represents $\hat{m}_0$ is also a random variable, say M, which depends on W, G, and $\hat{Y}$. Given the

triplet $(\mathcal{M}, W, \hat{Y})$, the joint probability that $M = j\varepsilon\mathcal{M}$ and $N = n$, given $Q = i\varepsilon\mathcal{H}$, denoted by

$$f_{M,N|Q}(j, n|i)(i, j=0, \ldots, \pm u; n = n_L, \ldots, n_U), \quad (9)$$

can be found using equations 8. Here n_L and n_U, the lower and upper bounds, respectively, on the number N of observations are finite for a given stopping rule δ because N^* is finite. Equation 9 is an expression for the probability that the experiment ends at level $j\varepsilon\mathcal{M}$ after n observations, given the experiment begins at level $i\varepsilon\mathcal{H}$. Let $\ell = 2u + 1$ be the total number of levels in $\mathcal{M}$. Denote by

$$a_{ij} = g_{M|Q}(j|i) = \sum_{n=n_L}^{n_U} f_{M,N|Q}(j, n|i), \quad (10)$$

the conditional probability that the experiment ends at level j given it begins at level i. Then $A = (a_{ij})$ is a ℓ x ℓ stochastic matrix which will be referred to as the Procedure Transition Matrix. Similarly, denote by

$$b_{ij} = g_{N|Q}(j|i) = \sum_{m=-u}^{u} f_{M,N|Q}(m, j|i), \quad (11)$$

the conditional probability that the experiment ends after j observations given it begins at level i. Then $B = (b_{ij})$ is a ℓ x $(n_U - n_L + 1)$ stochastic matrix which will be referred to as the Procedure Iteration Matrix. It will be shown below that the matrices A and B characterize procedures in $\mathcal{S}$.

B. A SPECIAL CASE

Consider the special cases $W(\delta, k_t, k_t^o, C_t) = W(\delta, k, k^o, 1)$ in $\mathcal{C}$. In dealing with these cases, it will be convenient to work with the random variable N*, the number of trials, rather than N the number of observations. Since $k_t = k$ for all t, N, which is easily obtained from (4), becomes

$$N = k\, N^*. \tag{12}$$

This change is made to permit easy comparison among procedures having a different number k of observations per trial. Since $C_t = 1$ for all t, all the levels in $\mathcal{M}$ are accessible from any level in $\mathcal{M}$ in a finite number of trials (that is, all the levels communicate with each other). Moreover, the probability that a particular level in $\mathcal{M}$ is selected for the $(t+1)^{th}$ trial depends only on the level used and the response obtained on the t-th trial. Hence, the stochastic process $\{Y_t;\ t \geq 1\}$ generated by $W(\delta, k, k^o, 1)$ has $\mathcal{M}$ as its state space, forms a random walk on $\mathcal{M}$ between reflecting barriers at levels -u and u and is thus a finite, irreducible Markov chain.[7]

In particular, the elements of the $\ell \times \ell$ one-step transition probability matrix $P = (p_{ij})$ are given by (7) where from (8)

[7] The terminology used in classifying the chain corresponds to that used by Cox and Miller [3] and Karlin [11].

$$
p_i = \begin{cases} 1 & i=-u \\ \sum_{j=0}^{k^o} \binom{k}{j} \alpha_i^{\,j} \beta_i^{\,k-j} & i=-u+1,\ldots,u-1 \\ 0 & i=u \end{cases}
$$

$$
r_i = \begin{cases} \sum_{j=k^o+1}^{k-k^o-1} \binom{k}{j} \alpha_i^{\,j} \beta_i^{\,k-j} & i=-u+1,\ldots,u-1 \\ 0 & i=\pm u \end{cases} \qquad (13)
$$

$$
q_i = \begin{cases} 0 & i=-u \\ \sum_{j=k-k^o}^{k} \binom{k}{j} \alpha_i^{\,j} \beta_i^{\,k-j} & i=-u+1,\ldots,u-1 \\ 1 & i=u \end{cases}
$$

and use has been made of equations 6 for $i = \pm u$. Thus, P has positive elements only on its principal diagonal and the diagonals just above and below the principal diagonal; that is, P is of the form

$$P = \begin{bmatrix} 0 & 1 & 0 & & & & & & \\ q_{-u+1} & r_{-u+1} & p_{-u+1} & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ & \cdot & \cdot & \cdot & & & & & \\ & & \cdot & \cdot & \cdot & & & & \\ & & & q_0 & r_0 & p_0 & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & & & & & q_{u-1} & r_{u-1} & p_{u-1} \\ & & & & & & & 0 & 1 & 0 \end{bmatrix}$$

where the $(\ell-2) \times (\ell-2)$ sub-matrix of P which excludes the first and last rows and columns of P is a matrix of Jacobi [8]. Furthermore, it follows from (5) and the symmetry of G that $\alpha_{-i} = \beta_i$; and thus $p_{-i} = q_i$, $r_{-i} = r_i$ and $q_{-i} = p_i$ $(i = 0, \pm 1, \ldots, \pm u)$. With these results the matrix P can be rewritten

$$
P = \begin{bmatrix}
0 & 1 & 0 & & & & & & \\
p_{u-1} & r_{u-1} & q_{u-1} & & & & & & \\
 & \ddots & \ddots & \ddots & & & & & \\
 & & p_0 & r_0 & p_0 & & & & \\
 & & & \ddots & \ddots & \ddots & & & \\
 & & & & q_{u-1} & r_{u-1} & p_{u-1} & & \\
 & & & & & 0 & 1 & 0
\end{bmatrix} \qquad (14)
$$

Note that P has the property that if the elements above the secondary diagonal are transposed, the resulting matrix is symmetric with respect to the secondary diagonal. This property will be referred to as reflective symmetry about the center element r_0. Note that if $k = 1$, then $k^0 = 0$ by definition and it follows from (8) that

$$r_i = P_r(0<J_t<1) = 0$$

for $i = 0, \pm 1, \ldots \pm(u-1)$. Thus when $k = 1$ the principal diagonal of P contains all zeros.

The stochastic process $\{Y_t;\ t \geq 1\}$ can now be further classified for $k>1$. To determine the periodicity of the chain, it is sufficient to consider the elements r_i in P and the level 0 [11]. When $k = 1$, $r_i = 0$ for all i, and the chain has period two since a return to the starting level is possible only in an even number of trials. When

$k>1$, $r_0>0$ and the chain is aperiodic since the process can reach level 0 and remain there indefinitely regardless of its initial level in $\mathcal{M}$. The chain is necessarily positive-recurrent because its state space $\mathcal{M}$ is finite; that is, the probability that the process returns to level i after a finite number of trials is one. Hence the stochastic process $\{Y_t;\ t\geq 1\}$ formed by $W(\delta, k, k^o, 1)$ is completely classified. For $k>1$, the process is a finite, irreducible, aperiodic Markov chain[8] and hence stationary probabilities can be found for the chain.

It was mentioned earlier that one may wish to compare procedures of the special case having a different number k, of observations per trial. With this in mind as an eventual objective, the asymptotic property of the P matrix as k increases is next considered. Note that (14) can be written

$$P = \begin{bmatrix} 0 & 1 & 0 & & & & \\ p_{u-1} & 1-p_{u-1}-q_{u-1} & q_{u-1} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & p_0 & 1-2p_0 & p_0 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & q_{u-1} & 1-p_{u-1}-q_{u-1} & p_{u-1} \\ & & & & 0 & 1 & 0 \end{bmatrix} \qquad (15)$$

[8] If a chain is both positive-recurrent and aperiodic it is said to be ergodic [3]. Kemeny and Snell [12] call a chain of this type a regular Markov chain.

and from (13), for $k^{o}<\infty$, it is seen that

$$\lim_{k\to\infty} p_i = \lim_{k\to\infty} q_i = 0 .$$

Hence,

$$\lim_{k\to\infty} P = \begin{bmatrix} \begin{matrix} 01 \\ 01 \end{matrix} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \begin{matrix} 10 \\ 10 \end{matrix} \end{bmatrix} = P^{*} \qquad (16)$$

where I is a $(\ell-4) \times (\ell-4)$ identity matrix. Intuitatively, this limit property implies that as the number k, of observations increase without bound, the probability that the process $\{Y_t;\ t\geq 1\}$ remains at its starting level is one, for $i = 0, \pm 1, \ldots, \pm(u-1)$. This can also be seen from (3) by noting that

$$\lim_{k\to\infty} \Pr\,(Y_{t+1} = Y_t) = \lim_{k\to\infty} \Pr\,(k^{o}<J_t<k - k^{o}) = 1.$$

In the two sections which follow, the properties of the A and B matrices introduced earlier will be considered. First, changes as a consequence of (12) will be noted. Then the asymptotic properties of the A and B matrices as k increases without bound will be considered for a specific example.

1. The A and B Matrices

The Procedure Transition and Iteration Matrices in the special cases are found in a manner analagous to that

used above. The resulting expressions can be obtained by replacing N by N* in (9)-(11). The i-th row of A is, as in the general case, the conditional probability that the experiment ends at level j given it begins at level i; while in this case, the i-th row of B is the conditional probability that the experiment ends after j trials given it begins at level i.

2. Properties of the A and B Matrices Given the Triplet $(\mathcal{M}, W, \hat{Y})$

As a result of the fact that the stochastic process $\{Y_t; t \geq 1\}$ generated by $W(\delta, k, k^o, 1)$ is a finite, irreducible Markov chain and the parameters k and k^o are constant for any particular W, the A and B matrices can be further studied to determine their asymptotic properties as the number k of observations per trial for a given triplet $(\mathcal{M}, W, \hat{Y})$.

Consider the set of procedures, Γ_1 in $\mathcal{C}$ defined by $W(\delta, k, k^o, 1) = W(\delta_1, k, 0, 1)$ where the rule δ_1 is to stop the experiment after both responses and nonresponses have been observed on two successive trials. For simplicity, let $\mathcal{M} = \{-2, -1, 0, 1, 2\}$. Then $\ell = 5$; $m = 1$; $n^*_L = 2$; and, since it is possible to start the experiment at level -2 and not obtain a response until level 2 is reached, $n^*_U = 5$. From (15) the one-step transition probability matrix P is

$$P = \begin{array}{c|ccccc} & -2 & -1 & 0 & 1 & 2 \\ \hline -2 & 0 & 1 & 0 & 0 & 0 \\ -1 & p_1 & 1-p_1-q_1 & q_1 & 0 & 0 \\ 0 & 0 & p_0 & 1-2p_0 & p_0 & 0 \\ 1 & 0 & 0 & q_1 & 1-p_1-q_1 & p_1 \\ 2 & 0 & 0 & 0 & 1 & 0 \end{array} \qquad (17)$$

where from (3), (8) and (13) for $k^{\circ} = 0$,

$$p_0 = \Pr(Y_{t+1} = \pm 1 | Y_t = 0) = \beta_0^{\,k}$$

$$p_1 = \Pr(Y_{t+1} = \pm 2 | Y_t = \pm 1) = \beta_1^{\,k}$$

$$q_1 = \Pr(Y_{t+1} = 0 | Y_t = \pm 1) = \alpha_1^{\,k}$$

$$\Pr(Y_{t+1} = \pm 1 | Y_t = \pm 2) = 1$$

To determine the probability $f_{M,N^*|Q}$ that the experiment ends at level M after N* trials given it started at level Q, it is convenient to first consider which combinations of levels and responses are required by $\hat{Y}$ to place $\hat{m}_0$ at a particular level in $\mathcal{M}$. Here, the estimator,

$$\hat{Y} = \frac{1}{2} \sum_{t=n^*-1}^{n^*} Y_t - \frac{m}{k} \sum_{t=n^*-1}^{n^*} J_t + m, \qquad (18)$$

from the elevation procedure of the field artillery precision registration technique (discussed further in IV) will be used. Note that the estimator (18) requires data from only the last two trials, the same number of trials

specified by the stopping rule; that is, the mean is computed from the data obtained from trials at two successive levels in $\mathcal{M}$ which result in both responses and nonresponses. Note also that in case there are k total responses, then (18) reduces to the mean level tested on the last two trials. Next, recall that the set of levels $\mathcal{M}$ represents the midpoints of a finite set of disjoint intervals of equal length m, where the level 0 (the mean of G) represents the midpoint of an open interval and the remaining intervals are half-open. Here, since m = 1, $\mathcal{M} = \{-2,\ldots,2\}$ represents the set of intervals,

$$\mathcal{I} = \left\{\left[\frac{-5}{2}, \frac{-3}{2}\right], \left(\frac{-3}{2}, \frac{-1}{2}\right], \left(\frac{-1}{2}, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{2}\right), \left[\frac{3}{2}, \frac{5}{2}\right)\right\} . \tag{19}$$

If $\hat{m}_0$ is in the interval $(-\frac{1}{2}, \frac{1}{2})$, then $\hat{m}_0$ is represented by level 0 and the random variable M is zero.

It will now be shown that sufficient conditions for determining the value of M can be given on the total number J_T of responses obtained on the last two trials. If the experiment starts at level 0 (that is, $Q = Y_1 = 0$) and $1 \leq J_1 \leq k - 1$, then it follows from (13) that $Y_2 = 0$ since $k^o = 0$. Also the experiment will end in this case after two trials regardless of the outcome of the second trial. From (18) one can determine upper and lower bounds on the total number of responses observed on the last two trials

$$J_T = \sum_{t=n^*-1}^{n^*} J_t,$$

which are sufficient to determine the level of $\hat{m}_o$ and thus, M. That is; for M = 0 it must be true that

$$\frac{-1}{2} < \hat{m}_0 < \frac{1}{2} .$$

If $Y_1 = 0$, and $Y_2 = 0$ then it follows from (18) and the fact that N* = 2 that

$$\frac{-1}{2} < 0 - \frac{J_T}{k} + 1 < \frac{1}{2}$$

which implies that

$$\left[\frac{k}{2}\right] < J_T < 2k - \left[\frac{k}{2}\right] \qquad (20)$$

where $\left[\frac{k}{2}\right]$ denotes the greatest integer in $\frac{k}{2}$. Similarly for M = -1 and M = 1, given $Y_1 = 0$ and $Y_2 = 0$, it follows that $2k - \left[\frac{k}{2}\right] \leq J_T \leq 2k - 1$ and $1 \leq J_T \leq \left[\frac{k}{2}\right]$, respectively. In a like manner sufficient conditions on J_T for determining M can be found for other outcomes of the experiment and conveniently displayed at the terminal branches a logic tree such as that shown at Figure 1 for Q = 0. It should be noted that the lower bound condition on J_T determines the upper bound for M and vice versa. For example, if $Y_1 = 0$ and $Y_2 = 0$, then using (18) and (20) one obtains for M = 0,

$$\max \hat{m}_0 = 0 - \frac{1}{k}\left(\frac{k}{2} + \varepsilon\right) + 1 = \frac{1}{2} - \frac{\varepsilon}{k}$$

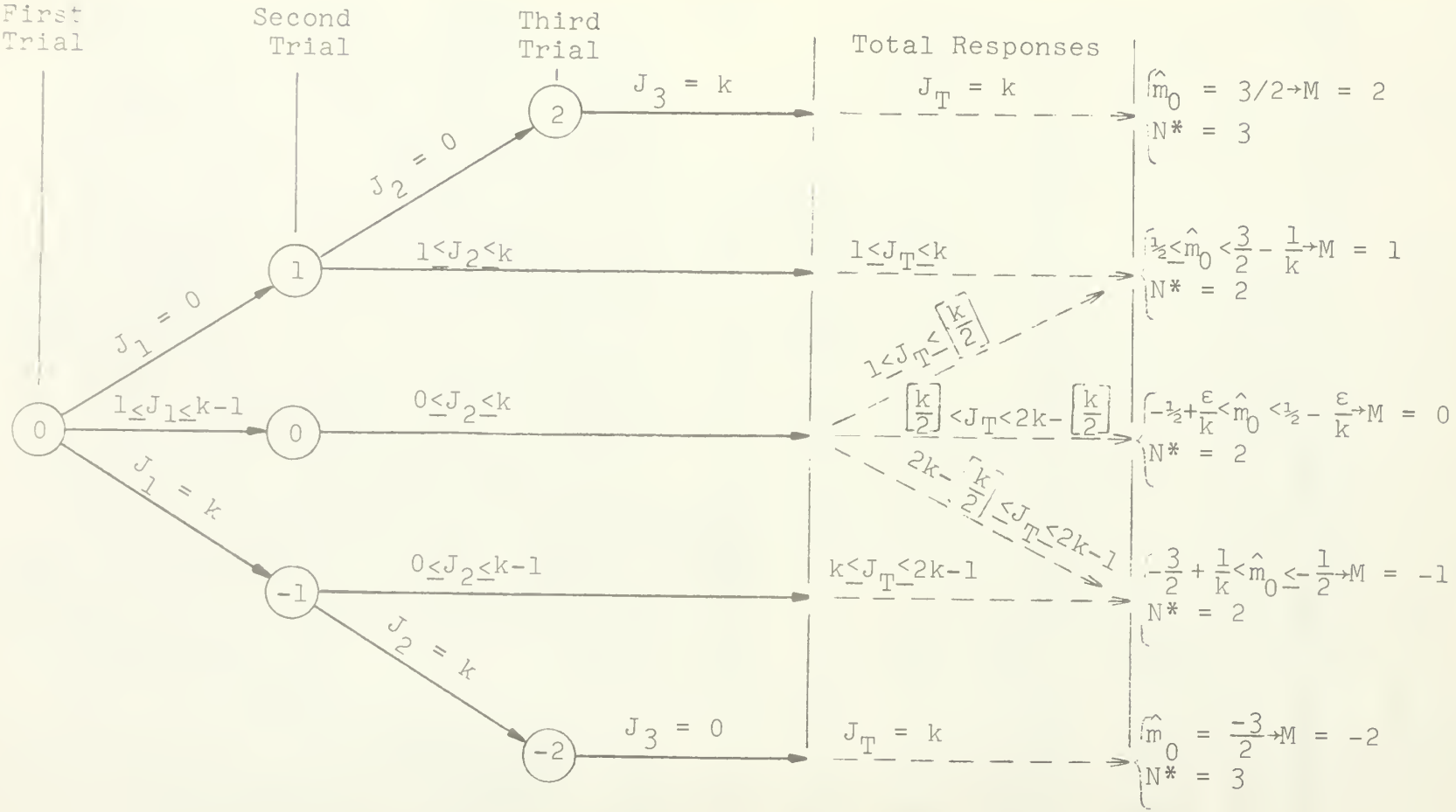

Figure 1. Logic Tree for $W(\delta_1, k, 0, 1)$ on $\mathcal{M} = \{-2, \ldots, 2\}$ Given the Experiment Begins at Level 0 (Here $\hat{Y}$ is given by equation 18).

since $\left\lceil \frac{k}{2} \right\rceil < \frac{k}{2} + \varepsilon < J_T$, for some $\varepsilon > 0$, and

$$\min \hat{m}_0 = 0 - \frac{1}{k}\left(2k - \frac{k}{2} - \varepsilon\right) + 1 = -\frac{1}{2} + \frac{\varepsilon}{k}$$

since $J_T < 2k - \frac{k}{2} - \varepsilon < 2k - \left\lceil \frac{k}{2} \right\rceil$, for some $\varepsilon > 0$. Note that the number N^*, of trials is readily determined for each of the tree branches.

The joint conditional probability $f_{M,N^*|Q}$ can be determined using diagrams such as Figure 1, the P matrix, and conditional probability arguments. Let $g_{i,j}$ be the probability that the total number of responses obtained on the last two trials satisfies the sufficient conditions for $M = j$ given that the last trial was at level i. In general, these probabilities are easily obtained except when two successive trials are conducted at the same level. For instance, from Figure 1 note that $\frac{3}{2} \leq \hat{m}_0 < \frac{5}{2}$, the condition for $M = 2$, is satisfied by the upper most branch of the tree; that is,

$$g_{2,2} = \Pr(J_T = k | J_2 = 0)$$

$$= \Pr(J_3 = k | J_2 = 0) = 1 . \qquad (21)$$

And when $M = 0$, it is seen that

$$g_{0,0} = \Pr\left(\left[\frac{k}{2}\right] < J_T < 2k - \left[\frac{k}{2}\right] \middle| 1 \le J_1 \le k-1\right)$$

$$= \Pr\left(\left[\frac{k}{2}\right] + 1 \le J_1 + J_2 \le 2k - \left[\frac{k}{2}\right] - 1 \middle| 1 \le J_1 \le k-1\right)$$

$$= \frac{\sum_{j=1}^{k-1} \Pr\left(\left[\frac{k}{2}\right] + 1 - j \le J_2 \le 2k - \left[\frac{k}{2}\right] - 1 - j \middle| J_1 = j\right)}{\Pr(1 \le J_1 \le k-1)}. \tag{22}$$

In a similar manner, one obtains:

$$g_{1,1} = \Pr(1 \le J_2 \le k | J_1 = 0) = 1 - p_1,$$

$$g_{0,1} = \frac{\sum_{j=1}^{k-1} \Pr(1 - j \le J_2 \le \left[\frac{k}{2}\right] - j | J_1 = j)}{\Pr(1 \le J_1 \le k-1)},$$

$$g_{0,-1} = \frac{\sum_{j=1}^{k-1} \Pr(2k - \left[\frac{k}{2}\right] - j \le J_2 \le 2k - 1 - j | J_1 = j)}{\Pr(1 \le J_1 \le k-1)}, \tag{23}$$

$$g_{-1,-1} = \Pr(0 \le J_2 \le k-1 | J_1 = k) = 1 - p_1,$$

$$g_{-2,-2} = \Pr(J_3 = 0 | J_2 = k) = 1.$$

Note that the $g_{i,i}$, $i \neq 0$, can be expressed in terms of the elements of the P matrix given by (17). Also, the denominator of the expressions for $g_{0,j}$, $j = -1, 0, 1$, is by (13), equal to r_0. These two relationships were also found to hold when $Q \neq 0$. Note also that $g_{0,1}$ and $g_{0,-1}$ are the tail probabilities of the conditional probability distribution of the random variable J_2 given the random variable J_1 is

not 0 or k and that $g_{i,i} = g_{-i,-i}$, i=1,2. Indeed, it can be shown from expressions similar to (20)-(23) that

$$g_{i,j} = g_{-i,-j}$$

for i,j = 0, ±1, ±2. Thus, the notation can be eased somewhat in the discussion that follows.

From the Markovian nature of the procedure being studied it follows that the joint conditional probability, $f_{M,N^*|Q}$, is now easily written for Q = 0 using equations (21) through (23), the elements of the P matrix given by (17), and Figure 1. That is;

$$f_{M,N^*|Q}(2,3|0) = g_{2,2}\cdot \Pr(Y_3=2|Y_{2=1})\cdot\Pr(Y_2=1|Y_1=0) = p_0p_1,$$

$$f_{M,N^*|Q}(1,2|0) = g_{1,1}\cdot \Pr(Y_2=1|Y_1=0)+g_{0,1}\cdot\Pr(Y_2=0|Y_1=0)$$
$$= p_0(1-p_1) + r_0g_{0,1},$$

$$f_{M,N^*|Q}(0,2|0) = g_{0,0}\cdot \Pr(Y_2=0|Y_1=0) = r_0g_{0,0},$$

$$f_{M,N^*|Q}(-1,2|0) = g_{0,1}\cdot \Pr(Y_2=-1|Y_1=0)+g_{0,1}\cdot\Pr(Y_2=0|Y_1=0)$$
$$= p_0(1-p_1) + r_0g_{0,1},$$

$$f_{M,N^*|Q}(-2,3|0) = g_{2,2}\cdot \Pr(Y_3=-2|Y_2=-1)\cdot\Pr(Y_2=-1|Y_1=0)=p_0p_1.$$

Note that

$$\sum_{j=-2}^{2} f_{M,N^*|Q}(j,n^*|0) = 2p_0 + r_0 \sum_{j=-2}^{2} g_{0,j} = 1,$$

since $\sum_j g_{0j} = 1$ and $r_0 = \begin{cases} 0 & k=1 \\ 1-2p_0 & k>1 \end{cases}$.

The probabilities, $f_{M,N^*|Q}(j,n^*|i)$ for $i \neq 0$, can be determined in a similar fashion so that from (10) and (11) the procedure transition and iteration matrices can be written:

$$A = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccccc} -2 & -1 & 0 & 1 & 2 \end{array} \\ \begin{bmatrix} p_1 & p_0q_1+r_1 & r_0q_1 & p_0q_1\,(1-p_1) & p_0q_1p_1 \\ p_1+r_1g_{1,2} & p_0q_1+r_1g_{1,1} & r_0q_1+r_1g_{1,0} & p_0q_1\,(1-p_1) & p_0q_1p_1 \\ p_0p_1 & p_0(1-p_1)+r_0g_{0,1} & r_0g_{0,0} & p_0(1-p_1)+r_0g_{0,1} & p_0p_1 \\ p_0q_1p_1 & p_0q_1\,(1-p_1) & r_0q_1+r_1g_{1,0} & p_0q_1+r_1g_{1,1} & p_1+r_1g_{1,2} \\ p_0q_1p_1 & p_0q_1\,(1-p_1) & r_0q_1 & p_0q_1+r_1 & p_1 \end{bmatrix} \end{array} \tag{24}$$

$$B = \begin{array}{c} \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1-q_1 & q_1\,(1-p_0) & p_0q_1\,(1-p_1) & p_0q_1p_1 \\ 1-p_0q_1 & p_0q_1\,(1-p_1) & p_0q_1p_1 & 0 \\ 1-2p_0p_1 & 2p_0p_1 & 0 & 0 \\ 1-p_0q_1 & p_0q_1\,(1-p_1) & p_0q_1p_1 & 0 \\ 1-q_1 & q_1\,(1-p_0) & p_0q_1\,(1-p_1) & p_0q_1p_1 \end{bmatrix} \end{array} . \tag{25}$$

Here, a_{ij} is the probability that the experiment ends at level j given it starts at level i and b_{ij} is the probability that the experiment requires j trials given it starts at level i. Note that A has the reflective symmetry property earlier defined for the P matrix and that B is symmetric with respect to its center row.

Next consider the asymptotic properties of A and B as k increases without bound. First note that the non-zero elements of each are exponential functions in k. Since p_i, q_i, and $g_{i,j}$ for $j \neq i$ are tail probabilities of binomially distributed variables and thus approach zero as k gets large, it is seen from (24) and (25) that

$$\lim_{k\to\infty} A = P^* = A^* \tag{26}$$

$$\lim_{k\to\infty} B = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix} = B^* \tag{27}$$

where P* is given by (16). The positive elements in A* will be referred to as the limiting "diagonal" of A. Intuitively, these results imply that as the number k, of observations per trial is increased; the level of the estimated mean j, becomes the starting level i, for all levels $i \in \mathcal{N}$ except $i = \pm u$, and the number of trials becomes the lower bound n_L^* of N*. Note that these implications are consistent with the interpretation given for P*.

In practice A* and B* are of little interest since there is usually a physical or monetary constraint on the number

of observations which can be taken. A property which may be of interest however is the rate at which A and B approach their limiting forms. If

$$A_1(k) = (\Delta_k \, a_{ij}\,(k)), \tag{28}$$

$$B_1(k) = (\Delta_k \, b_{ij}\,(k)), \tag{29}$$

where $\Delta_k \, x_{ij}(k)$ is the receeding difference of $x_{ij}(k)$ with respect to k

$$\Delta_k \, x_{ij}(k) = x_{ij}(k) - x_{ij}\,(k-1)$$

then it can be shown that

$$\lim_{k\to\infty} \; A_1 = 0 \tag{30}$$

$$\lim_{k\to\infty} \; B_1 = 0 \; . \tag{31}$$

Furthermore, there exists a k′ such that for k>k′ the elements of A on the limiting diagonal are increasing to one while all the other elements of A are decreasing to zero. Likewise, the elements of the first column of B are strictly increasing to one while all other elements are at zero or decreasing to zero.

While the results of this section hold specifically for procedures Γ_1 in $\mathcal{C}$ and the estimator defined by (18), it is believed that similar results can be obtained for other procedures in $\mathcal{C}$ defined by $W(\delta_i, k, 0, 1)$. That is, a change in the stopping rule δ_1 to permit more trials,

should not alter the result that as k increases the level of the estimated mean approaches the starting level of the experiment and that the number of trials approaches the lower bound permitted by the stopping rule.

In a later section, the efficiency of alternative procedures in Γ_1 will be examined at which time there will be occasion to refer to some of the results obtained above.

III. EFFICIENCY OF THE GENERAL PROCEDURE

In this chapter, a methodology for evaluating alternative procedures in the class $\mathcal{C}$ will be discussed. A loss plus cost objective function is used with an appropriate criterion to determine an optimal procedure in a manner analogous to Wald's decision problem [17] where the space of terminal decisions is $\mathcal{M}$, the finite set of equally spaced levels. No attempt will be made to determine the optimal procedure in $\mathcal{C}$, rather it is assumed that a finite number of procedures in $\mathcal{C}$, say Γ, are to be evaluated and compared.

Recall, an experiment ends at some level M in $\mathcal{M}$ when $\hat{m}_o$, determined by use of $\hat{Y}$, is at level M. Suppose there exists some procedure W* in $\mathcal{C}$, not necessarily in Γ the set of procedures being evaluated, whose use will almost surely cause the experiment to end at level 0, the level of the true mean. Then W* is preferred to a procedure which results in the experiment ending at level j, $j \neq 0$.

Let $H = h(j)(j = 0, \pm 1, \ldots, \pm u)$ be an integer valued function which gives the loss incurred when the experiment ends at level j. Note that h(M) is a random variable. Since it is preferred that the experiment end at level 0, the loss h(0) can be taken as zero. The structure of the loss function may be known or its form assumed. Indeed, the nature of the random variable having the distribution

G or the intended use of $\hat{m}_0$, the estimated mean, may suggest an appropriate structure. If the structure of the loss function is unknown, one of several empirically derived loss functions might be used. For instance, one which is frequently cited in the literature is quadratic in the distance between the estimate $\hat{m}_0$ and the true mean m_0 [17]. In this case,

$$h(j) = \eta(j-0)^2 = \eta j^2, \tag{32}$$

where η is an integer weighting factor, might be used. Suppose the cost of experimentation is known; that is, there is some constant cost, called the "set-up" cost which is independent of the number of observations taken, and some variable cost, dependent upon the number of observations taken, both of which are known. Without loss of generality define h(j), often called the cost of missclassification, in units of the cost of observations and take the "set-up" cost as zero. Then the total loss incurred L, is given by:

$$L = H + N, \tag{33}$$

where L is also in units of the cost of an observation.

In the next two sections the problem of choosing an optimal procedure W* from a set of alternatives Γ is considered, first using criteria based upon the expected value of the total loss and then using criteria based upon the variance of the total loss. Finally, the results of the

chapter are extended to the special case discussed in the preceeding section.

A. CRITERIA USING THE EXPECTED VALUE OF THE LOSS

Given a particular procedure, W in Γ, if an experiment is performed a large number of times and the loss evaluated by (33); then the expected value and variance of the average total loss, $\mathcal{L}_\nu$, are given by:

$$E[\mathcal{L}_\nu] = E[L]$$

$$\text{Var}[\mathcal{L}_\nu] = \frac{1}{\nu} \text{Var}[L],$$

where ν is the number of replications of the experiment [13]. These results express the principle of the law of large numbers in that as ν increases to infinity the variance of the average total loss goes to zero. Furthermore, the average total loss $\mathcal{L}_\nu$ is approximated by the expectation of the total loss, given by

$$E[L] = E[H] + E[N]. \tag{34}$$

An expression for (34), the expected total loss, which depends upon the prior distribution of Q, and A and B matrices discussed in the previous chapter, the loss function h and the range of N will now be derived.

Let F in Ω be the distribution function of the random variable Q with mass function f(i) denoting the probability that the experiment starts at level i ($i = 0, \pm 1, \ldots, \pm u$). Here Ω is a set of distribution functions, the elements of

which, in general, are not completely specified. Let $\bar{q}$ in Ω be a ℓ x 1 column vector whose i-th component is a prior estimate of f(i); that is, $\bar{q}$ represents a prior distribution on Q. Then the mass function of the random variable M is given by

$$g_M(j) = \bar{q}' A; \tag{35}$$

that is, $g_M(j)$ is the probability that the experiment ends at level j ($j = 0, \pm 1, \ldots, \pm u$). If $\bar{h}_1$ is an ℓ x 1 column vector whose j-th component is the loss h(j) incurred when the experiment ends at level j, then

$$E[H] = \sum_{j=-u}^{u} g_M(j)\, h(j) = \bar{q}' A \bar{h}_1 . \tag{36}$$

Similarly, the mass function of the random variable N is given by:

$$g_N(n) = \bar{q}' B, \tag{37}$$

where $g_N(n)$ is the probability that the experiment ends after n observations ($n = n_L, \ldots, n_U$). If $\bar{n}_1$ is a $(n_U - n_L + 1)$ x 1 column vector,

$$\bar{n}_1 = \begin{bmatrix} n_L \\ \vdots \\ n_U \end{bmatrix},$$

then the expected number of observations is given by:

$$E[N] = \sum_{n=n_L}^{n_U} g_N(n) \cdot n = \bar{q}' B \bar{n}_1. \tag{38}$$

Hence, using (36) and (38), the expected total loss given in (34) can be written

$$E[L] = \bar{q}' (A\bar{h}_1 + B\bar{n}_1) = \bar{q}' \bar{r}; \tag{39}$$

where

$$\bar{r} = A\bar{h}_1 + B\bar{n}_1 = E[L/Q] \tag{40}$$

is a ℓ x 1 vector of conditional expected total losses. Note that the components of $\bar{r}$ depend upon W. In fact (40) represents, using Wald's notation [17], a vector valued function

$$\bar{r} = \bar{r}(W) \tag{41}$$

the i-th component of which is the expected total loss (risk) given the experiment starts at level i; i = 0, $\pm 1, \ldots, \pm u$. Likewise the expected total loss is equivalent to Wald's average risk function [17]; that is,

$$E[L] = \bar{q}' \bar{r} = r^*(\bar{q}, W). \tag{42}$$

Note that in case there is no loss associated with an experiment that ends at level j (that is, h (j) = 0, for all j), then (39) reduces to the expected number of observations given by (38). Given a prior distribution on Q, it is clear that the expected loss may be determined from (39) for each procedure W in Γ. The remainder of this section is devoted to a consideration of decision criteria that might be used in finding the best procedure in Γ.

1. Bayes Procedure

A procedure W* in Γ which minimizes the expected total loss over all W in Γ may be regarded as an optimal procedure [17]. More precisely a procedure W* such that

$$E\,[L] = r^*\,(\bar{q},W^*) = \min_{W\varepsilon\Gamma} r^*\,(\bar{q},W), \tag{43}$$

is called a Bayes procedure in Γ relative to $\bar{q}$. Thus, if prior information on the distribution F exists and is known to the experimenter, he may evaluate alternative procedures in the set Γ using the Bayes criterion as a measure of the optimal procedure's efficiency. Such conditions may arise, for example, as a result of pilot testing. Indeed, in the elevation procedure of the field artillery precision registration technique, discussed in IV, it is often possible to obtain such information. Note that in case the distribution F is such that the experiment starts at some level i with probability one, that is, if

$$\bar{q} = \bar{e}_i \tag{44}$$

where $\bar{e}_i$ is a ℓ x 1 column vector having one as its i-th component and zero elsewhere, then (39) becomes

$$E\,[L] = \bar{e}_i' \bar{r} = r_i \quad (i=0, \pm 1,\ldots, \pm u).$$

The set of vectors defined by (44) is called the set of unit vectors in Euclidean ℓ-space and will be denoted $\bar{q} \varepsilon E^{\ell}$.

2. Minimax Procedure

A procedure W^* in Γ such that

$$E[L] = r^*(\bar{q}^*, W^*) = \min_{W \in \Gamma} \max_{\bar{q} \in \Omega} r^*(\bar{q}, W) \qquad (45)$$

is called a minimax procedure [17]. In some circumstances, particularly if the prior distribution on Q is not known, a minimax procedure may be regarded as an optimal procedure. Here one first determines for each W a vector $\bar{q}$ (a distribution) which is most unfavorable (that is, results in the largest expected loss) and then chooses W^* to minimize the effect of the unfavorable distributions.

B. CRITERIA USING THE VARIANCE OF THE LOSS

Now suppose some procedure W in Γ is known to be optimal with respect to one of the criteria discussed in the previous section. In some circumstances, W may be the optimal procedure for performing the experiment. But in other circumstances, it may be useful to know the variance of the total loss,

$$\text{Var}[L] = \text{Var}[H] + \text{Var}[N] + 2\,\text{Cov}[H, N] , \qquad (46)$$

where Cov [H, N] is the covariance of the random variables H and N, before the "best" procedure can be selected. An expression for the variance of the total loss, similar to the matrix equation given by (39) for the expected total loss, will now be derived.

Let $\bar{h}_2$ be a ℓ x 1 column vector whose j-th component is $[h(j)]^2$; that is, the square of the loss incurred when the experiment ends at level j ($j = 0, \pm 1, \ldots, \pm u$). Let $\bar{n}_2$ be a $(n_U - n_L + 1)$ x 1 column vector,

$$\bar{n}_2 = \begin{bmatrix} (n_L)^2 \\ \cdot \\ \cdot \\ (n_U)^2 \end{bmatrix}.$$

It follows that the second moments of H and N, respectively, are:

$$E[H^2] = \sum_{j=-u}^{u} g_M(j) [h(j)]^2 = \bar{q}' A \bar{h}_2, \tag{47}$$

$$E[N^2] = \sum_{n=n_L}^{n_L} g_N(n) n^2 = \bar{q}' B \bar{n}_2, \tag{48}$$

where use is made of (35) and (37). Hence the variance of H, using (47) and (36), can be written in matrix notation

$$\begin{aligned} \text{Var}[H] &= E[H^2] - E^2[H] \\ &= \bar{q}' A\bar{h}_2 - \bar{q}' A\bar{h}_1\bar{h}_1' A'\bar{q} \quad . \end{aligned} \tag{49}$$

Likewise, the variance of N can be written as

$$\begin{aligned} \text{Var}[N] &= E[N^2] - E^2[N] \\ &= \bar{q}' B\bar{n}_2 - \bar{q}' B\bar{n}_1\bar{n}_1' B'\bar{q}, \end{aligned} \tag{50}$$

upon substitution of (48) and (38).

The covariance of the random variables H and N is given by

$$\text{Cov}\,[H, N] = E\,[HN] - E\,[H] \cdot E\,[N], \tag{51}$$

where the second term on the right hand side of is known; that is, using (36) and (38) the product of the expected values of H and N can be written in matrix notation,

$$E\,[H] \cdot E\,[N] = \bar{q}' A \bar{h}_1 \bar{n}_1' B' \bar{q}. \tag{52}$$

The product moment E [HN] can be found from the joint conditional probability mass function of the random variables M and N, given the random variable Q. Recall $f_{M,N|Q}(j,n|i)$ gives the probability that the experiment ends at level j after n observations, given it starts at level i. It follows that the conditional product moment of H and N, given Q can be written:

$$E\,[HN|Q = i] = \sum_{j=-u}^{u} \sum_{n=n_L}^{n_L} n \cdot f_{M,N|Q}(j,n|i) \cdot h(j), \tag{53}$$

for $i = 0, \pm 1, \ldots, \pm u$. Denote by $C = (c_{ij})$ a $\ell \times \ell$ matrix, called the Product Moment Matrix, where:

$$c_{ij} = \sum_{n=n_L}^{n_U} n \cdot f_{M,N|Q}(j, n|i) \quad ,$$

then the conditional product moment of H and N, given Q is the scalar product of the i-th row of C, denoted C_i, and the vector $\bar{h}_i$. That is, equation 53 becomes

$$E\,[HN|Q = i] = C_i \bar{h}_i,$$

and it follows that

$$E\,[HN] = \sum_{i=-u}^{u} \Pr\,(Q=i) \cdot C_i \bar{h}_1 = \bar{q}' C\bar{h}_1. \qquad (54)$$

Thus, (51), the covariance of H and N, can be written

$$\mathrm{Cov}\,[H, N] = \bar{q}' C\bar{h}_1 - \bar{q}' A\bar{h}_1 \bar{n}_1' B'\bar{q} \qquad (55)$$

The variance of the total loss, equation 46, can now be written in matrix notation using (49), (50), and (55). After some rearrangement of terms one obtains

$$\begin{aligned}\mathrm{Var}\,[L] &= \bar{q}' (A\bar{h}_2 + B\bar{n}_2 + 2C\bar{h}_1) \\ &\quad - \bar{q}' (A\bar{h}_1\bar{h}_1' A' + B\bar{n}_1\bar{n}_1' B' + 2A\bar{h}_1\bar{n}_1' B')\bar{q} \\ &= \bar{q}' \bar{c} - \bar{q}' D\bar{q}, \qquad (56)\end{aligned}$$

where

$$\bar{c} = Ah_2 + Bn_2 + 2Ch_1$$

is a ℓ x 1 vector and

$$\begin{aligned}D &= A\bar{h}_1\bar{h}_1' A' + B\bar{n}_1\bar{n}_1' B' + 2A\bar{h}_1\bar{n}_1' B' \\ &= (A\bar{h}_1 + B\bar{n}_1)(A\bar{h}_1 + B\bar{n}_1)' \\ &= \bar{r}\,\bar{r}' \qquad (57)\end{aligned}$$

is a ℓ x ℓ matrix and r is the ℓ x 1 vector of conditional expected total losses defined by (40).

From (56) and (57), it is seen that the variance of the total loss is easily determined for a particular procedure W in Γ given a prior distribution on Q and the $\bar{r}$ vector.

But again one is confronted with the problem of choosing an appropriate criterion. Two criteria which may be used with (56) are considered below.

1. Minimum Variance Procedure

A procedure W* in Γ such that

$$\text{Var } [L] = v^* (\bar{q}, W^*) = \min_{W \in \Gamma} v^* (\bar{q}, W) \qquad (58)$$

will be called a minimum variance procedure in Γ relative to $\bar{q}$. Here, perhaps a certain amount of liberty has been taken in that (58) is defined analogous to (43) and thus W* might appropriately be called a Bayes variance procedure. An analogy also exists with minimum variance estimators in statistics which appears to be more related in the context used here, so that the first mentioned terminology will be used. Hence, if a prior distribution exists and is known to the experimenter, he may, as when working with the Bayes criterion, evaluate alternatives in Γ using (58) as a measure of the optimal procedure's efficiency. Note that if $\bar{q}$ is a unit vector as defined by (44), then (56) can be written

$$\text{Var } [L] = \bar{e}_i' \bar{c} - \bar{e}_i' D \bar{e}_i$$

$$= c_i - d_{ii} \quad (i=0, \pm 1, \ldots, \pm u).$$

2. Minimax Variance Procedure

Consider the variance of the total loss given by equation 56 above. It is known [10] that (56) is convex

or concave depending upon whether the matrix of the quadratic form, -D, is positive semidefinite or negative semidefinite. Hence, it must first be established that the matrix D is positive semidefinite before discussing maximization of (56) with respect to $\bar{q}$.

In (57), it is seen that $D = \bar{r}\,\bar{r}'$. Since $\bar{r}$ is a ℓ x 1 vector, its rank is $1 < \ell$ and it follows that D is a positive semidefinite, symmetric matrix, and is singular [9]. Hence, -D is negative semidefinite and the quadratic form given by (56) is concave.

Now consider the quadratic programming problem of finding a $\bar{q}^*$ which

$$\begin{aligned} &\text{maximizes} && \bar{q}'\,\bar{c} - \bar{q}'\,D\bar{q} \\ &\text{subject to} && \bar{q}'\,\bar{1} = 1 \\ & && \bar{q} \geq \bar{0} \end{aligned} \tag{59}$$

where $\bar{0}$ is the zero vector and $\bar{1} = (1,\ldots, 1)'$. Since the set of feasible solutions form a convex set and the objective function is concave, it is known from the theory of convex functions [10] that any relative maximum is also a global maximum. (In this case the global maximum is not unique since the quadratic form is not strictly concave.) Thus if a feasible solution to (59) exists, one may find a prior distribution $\bar{q}^*$ which maximizes the variance of the total loss, given by (56). Note that if $\bar{q}$ is a unit vector, then the solution to (59) may involve integer programming techniques since it must be true that $q_i = 1$,

$q_j = 0$ for all $j \neq i$ to satisfy the constraints of the problem.

An additional criterion may now be defined. A procedure W* in Γ such that

$$\text{Var } [L] = v^* (\bar{q}^*, W^*) = \min_{W \in \Gamma} \max_{\bar{q} \in \Omega} (\bar{q}' \bar{c} - \bar{q}' D \bar{q}) \qquad (60)$$

will be called a minimax variance procedure provided a feasible solution to (59) exists.

For a detailed discussion of techniques which can be used in solving (59) the reader is referred to Hadley [10]. It should be noted that maximization of (59) may be facilitated by the transformation

$$\bar{q} = R' \bar{z}$$

where R is a ℓ x ℓ non-singular matrix such that

$$RDR' = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

The reader is referred to Anderson [1] for proof that R exists and a method for its construction.

C. THE SPECIAL CASE

In the special case, development of the expressions for the total loss, the expected total loss and the variance of the total loss follow arguments similar to those given

above for the general case. The arguments are not repeated here; rather differences in the resulting expressions as a consequence of (12) are noted.

The total loss is given by

$$L = H + kN^*. \tag{61}$$

If n_1^* is a $(n_U^* - n_L^* + 1) \times 1$ column vector

$$\bar{n}_1^* = \begin{bmatrix} n_L^* \\ \cdot \\ \cdot \\ \cdot \\ n_U^* \end{bmatrix},$$

and $\bar{n}_2^*$ is a $(n_U^* - n_L^* + 1) \times 1$ column vector

$$\bar{n}_2^* = \begin{bmatrix} (n_L^*)^2 \\ \cdot \\ \cdot \\ \cdot \\ (n_U^*)^2 \end{bmatrix},$$

then the expressions for the expected value and variance in the number of trials can be obtained by substituting $\bar{n}_1^*$ for $\bar{n}_1$ and $\bar{n}_2^*$ for $\bar{n}_2$ in (38) and (50), respectively. The expected total loss is

$$E[L] = \bar{q}' (A\bar{h}_1 + kB\bar{n}_1^*) = \bar{q}'\bar{r}. \tag{62}$$

And the variance of the total loss is

$$\text{Var}[L] = \text{Var}[H] + k^2 \text{Var}[N^*] + 2k \text{Cov}[H, N^*],$$

$$= \bar{q}' (A\bar{h}_2 + k^2 B\bar{n}_2^* + 2kC\bar{h}_1)$$

$$- \bar{q}' (A\bar{h}_1\bar{h}_1' A' + k^2 B\bar{n}_1^*\bar{n}_1^{*\prime} B' + 2kA\bar{h}_1\bar{n}_1^{*\prime} B')\bar{q}$$

$$= \bar{q}' \bar{c} - \bar{q}' D\bar{q} \qquad (63)$$

where in this case the covariance of H and N* is obtained from (53) - (55), substituting $\bar{n}_1^*$ for $\bar{n}_1$,

$$\bar{c} = A\bar{h}_2 + k^2 B\bar{n}_2^* + 2kC\bar{h}_1,$$

$$D = (A\bar{h}_1 + kB\bar{n}_1^*)(A\bar{h}_1 + kB\bar{n}_1^*)'.$$

Obviously, one of the criteria defined by (43), (45) or (58), (60) can also be used in the special case. The problem, once one of these criteria has been selected, reduces to one of evaluating (62) or (63) for all procedures under consideration and then selecting the procedure which best satisfies the chosen criterion.

In the remaining paragraphs of this section, the specific example introduced in Chapter II will be continued. First, it will be shown that the Bayes procedure in the set Γ_1 can be found by analytical methods. Then an evaluation of some of the procedures in Γ_1 will be discussed.

1. Finding the Bayes Procedure Given the Triplet $(\mathcal{M}, W, \hat{Y})$

Consider the set of procedures Γ_1 defined by $W(\delta_1, k, 0, 1)$ on $\mathcal{M} = \{-2, -1, 0, 1, 2\}$ introduced in II B 2. Recall the rule δ_1 is to stop the experiment after

both responses and nonresponses have been obtained at two successive levels and that $N^* \varepsilon \{2, \ldots, 5\}$.

For this set of procedures, the experimenter might like to know if there exists a number k^*, of observations per trial which minimizes the expected total loss given by (62) for a given prior distribution and loss structure. For example, he may be faced with a constraint on the number of trials for the experiment rather than on the total number of observations. Such a situation may arise, for example, when there is a significant time lapse before response data can be observed. At any rate, a procedure with k^* observations per trial is a Bayes procedure. To see this, note from (62) and (42) that

$$E[L] = \bar{q}' (A\bar{h}_1 + kB\bar{n}_1^*) = r^* (\bar{q}, W).$$

Since k is the only unknown in $W(\delta_1, k, 0, 1)$, it follows from (43) that the procedure $W(\delta_1, k^*, 0, 1)$ is a Bayes procedure. The experimenter may also be interested in knowing if there is a relationship between the number of observations per trial and magnitude of the loss incurred from making a wrong decision. For example, if it is possible that the loss might be higher than he anticipates, he may be interested in knowing whether he should take a larger or smaller number of observations per trial.

For definiteness, suppose that the loss function is the quadratic function given in (32). Then the expected total loss can be written

$$E[L] = \bar{q}' (\eta A\bar{h}_1 + kB\bar{n}_1^*) \tag{64}$$

where

$$\bar{h}_1 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \quad \bar{n}_1^* = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \tag{65}$$

and A and B are given by (24) and (25) respectively and η is a loss parameter which determines the magnitude of the loss. The Bayes procedure may be found by minimizing (64) with respect to $k \geq 1$.

Since it is assumed that $\bar{q}$, the prior distribution of Q is given, it is sufficient to consider minimization of the components of the vector $\bar{r}$, of conditional expected loss,

$$\bar{r} = \bar{r}(k, \eta) = \eta A\bar{h}_1 + kB\bar{n}_1^* \tag{66}$$

Note that the components $\bar{r}$ can be made arbitrarily large for a given η by choosing k large; that is

$$\lim_{k\to\infty} \bar{r} = \lim_{k\to\infty} \left\{ \eta \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}, \tag{67}$$

where use has been made of (26), (27) and (65). Next, consider minimization of $\bar{r}$. Differencing (66) with respect to k yields

$$\Delta_k \bar{r}\ (k, \eta) = \eta A_1(k)\ \bar{h}_1 + kB_1(k)\ \bar{n}_1^* + B(k-1)\ \bar{n}_1^* \qquad (68)$$

where $A_1(k)$ and $B_1(k)$ are defined by (28) and (29) respectively. It can be shown that if $\eta = 0$, then

$$\Delta_k \bar{r}\ (k, 0) > 0,$$

for all $k \geq 1$; and if $\eta > 0$, then the existence of $k^* > 1$ which minimizes $\bar{r}$ depends on the magnitude of η. Thus, in this example, the conditional expected loss is minimized by taking $k = 1$ when $\eta = 0$ or by finding $k^* > 1$ if η is sufficiently large. For $k > k^*$, $\bar{r}$ is increasing in k. In fact, here

$$\lim_{k\to\infty} \Delta_k \bar{r}\ (k, \eta) = \lim_{k\to\infty} B\ (k-1)\ \bar{n}_1^* = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix},$$

where use is made of (30), (31), and the fact that $\lim_{k\to\infty} kB_1(k) = 0$. Given that $Q = i$ and η is sufficiently large, k^* is that k which satisfies,

$$\Delta_k r_i\ (k, \eta) < 0$$

$$\Delta_k r_i\ (k + 1, \eta) > 0.$$

Now consider the change in (68) as η changes

$$\Delta^2_{k,\eta}\ \bar{r}\ (k, \eta) = A_1(k)\ \bar{h}_1.$$

It can be shown that there exists a k' such that for $k > k'$,

$$\Delta^2_{k,\eta}\ \bar{r}\ (k, \eta) < 0,$$

a result which implies that the expected loss can be decreased for high loss structures by increasing the number of observations per trial.

It follows from the above that for the set Γ_1 of procedures defined by $W(\delta_1, k, 0, 1)$ and a quadratic loss function that the existence of an optimal number k^*, of observations per trial depends only on the magnitude of the loss which may be incurred from deciding that the mean of the distribution being sampled is at some level other than zero. That is, if there is no loss incurred by this decision, then one observation per trial should be taken; otherwise more than one observation probably should be taken. Also, it was shown that choosing W in Γ_1 by finding a number k^*, of observations per trial which minimizes the expected total loss is equivalent to finding a procedure which satisfies (43), the Bayes criterion.

Comparing procedures in the set Γ_1, discussed above, with Tsutakawa's procedures has been avoided because of the limitation imposed by the stopping rule δ_1, a rule selected because of its simplicity and applicability to the elevation procedure of the field artillery precision registration discussed in Chapter IV. Using the rule δ_1, it is noted that the up-and-down method (that is, $k = 1$) does not perform well when compared to procedures in Γ_1 with $k > 1$. It will be seen in IV that its performance is considerably better under a more liberal stopping rule.

Indeed, a limitation in evaluating procedures of the special case is the fact that comparisons are made over procedures where the total number of observations (sample size) varies drastically. As mentioned earlier, this situation may be realistic if the experimenter is faced with a constraint on the number of trials rather than the total number of observations.

It appears that, in the special case, whether or not there exists a number $k^* > 1$ of observations per trial which minimizes the expected total loss depends upon the nature of loss function and the stopping rule, δ. While the results of this section hold specifically for the stopping rule δ_1 and the estimator defined by (18) it is anticipated that similar results should be obtained for stopping rules which permit a larger number of trials and other estimators.

2. An Evaluation of Some Procedures in Γ_1

In order to illustrate the techniques of evaluation which have been discussed, it was decided to evaluate the loss characteristics for some procedures in Γ_1 using data generated by a high-speed didgital computer. Recall that Γ_1 is the set of procedures considered in the example introduced in section II B 2 and continued in the preceeding section. A computer program (see page 127) was written in FORTRAN IV and ran on an IBM 360 computer for $k = 1,\ldots, 6$, two loss functions, four values of the loss parameter, and

the set of prior distributions $\bar{e}_i$ in E^5, $i = 0, \pm 1, \pm 2$. Output from the program is arranged as follows: for each value of the number k of observations per trial, the A and B matrices are followed by matrices showing the expected value and variance of the total loss given $\bar{q} = \bar{e}_i$ in E^5, $i = 0, \pm 1, \pm 2$ (see page 100). In the remainder of this section, the results of the evaluation will be discussed.

Tables I and II show the optimal procedures in Γ_1 (that is, optimal number k, of observations per trial) for the criteria based upon the expected value and variance of the total loss, respectively, when the loss function is either linear or quadratic and the loss parameter, η, has the value indicated. In Table I, note that when $\eta = 0$, $k = 1$ is optimal; but when $\eta > 0$, $k > 1$ is optimal. Note also that using criteria based upon expected loss, the ordering among alternative procedures is independent of the two loss structures used for the values of η considered. On the other hand, from Table II, it is seen that use of variance criteria is very sensitive to the loss structure. For linear loss structures, variance criteria tend to indicate fewer observations per trial when compared to expected loss criteria and vice versa for quadratic loss structures. In general, the minimax criterion indicates optimality for procedures with a smaller number of observations per trial than the Bayes criterion relative to starting at level 0. In other words, as anticipated the minimax procedure was found to be protective against a high loss resulting from

TABLE I

OPTIMAL PROCEDURES (NUMBER OF OBSERVATIONS PER TRIAL)
USING EXPECTED LOSS CRITERIA

Loss Function	Bayes Procedures				Minimax Procedures			
	Value of Loss Parameter				Value of Loss Parameter			
	0	10	20	30	0	10	20	30
Linear	1	2(±2) 3(0,±1)	3(±1,±2) 5(0)	3(±2) 5(0,±1)	1	2	3	3
Quadratic	1	2(±2) 3(0,±1)	3(±1,±2) 5(0)	3(±2) 5(0,±1)	1	2	3	3

NOTE: The notation 2(±2) means that a procedure with two observations per trial is optimal relative to the prior distributions $\bar{e}_i \in E^5$, for $i = \pm 2$.

TABLE II

OPTIMAL PROCEDURES (NUMBER OF OBSERVATIONS PER TRIAL)
USING VARIANCE CRITERIA

Loss Function	Minimum Variance Procedures				Minimax Variance Procedures			
	Value of Loss Parameter				Value of Loss Parameter			
	0	10	20	30	0	10	20	30
Linear	1(±2) 5(0) 6(±1)	1(0) 6(±1,±2)	1(0,±2) 5(±1)	1(0,±2) 5(±1)	1	1	1	1
Quadratic	1(±2) 5(0) 6(±1)	5(0) 6(±1,±2)	5(0,±1) 6(±2)	5(0,±1) 6(±2)	1	5	6	6

NOTE: The notation 1(±2) means that a procedure with one observation per trial is optimal relative to the prior distributions $\bar{e}_i \in E^5$, for $i = \pm 2$.

starting away from level 0. On the other hand, it seems more difficult to explain the relation between the minimax variance criterion and the minimum variance criterion. For a linear loss structure, the minimax variance criterion indicates optimality for a procedure with one observation per trial which, in general, is also indicated by the minimum variance criterion relative to starting at level 0. But for a quadratic loss structure, the minimax variance criterion, in general, was found to be protective against high variances resulting from starting away from level 0.

Recall that the technique used in the preceeding section to show how the Bayes procedure could be found analytically consisted of minimizing (66) with respect to k. Figures 2 and 3 show the expected value of the loss, given the experiment begins at level 0, as a function of k and η for quadratic and linear loss structures. Note the increases which occur at k = 4. These anomalies are a consequence of (19); that is, the fact that level 0 is by definition the midpoint of the open interval $(-\frac{1}{2}, \frac{1}{2})$.[9] Note: the minima indicated by the curves correspond to the optimal Bayes procedures relative to $\bar{e}_0$ in E^5 which are indicated in Table I; the curves for $\eta = 0$ are strictly increasing; the curves are

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[9]It turns out that $g_{0,1}$, given by (23), is significantly higher for k = 4 than it is for k = 5 (approximately 0.12 as compared to 0.06). Thus if k = 4 and the experiment starts at level 0 there is a higher probability, relative to k = 3 or 5, that the experiment will end at some level other than level 0. The resulting effect is a higher expected loss for k = 4.

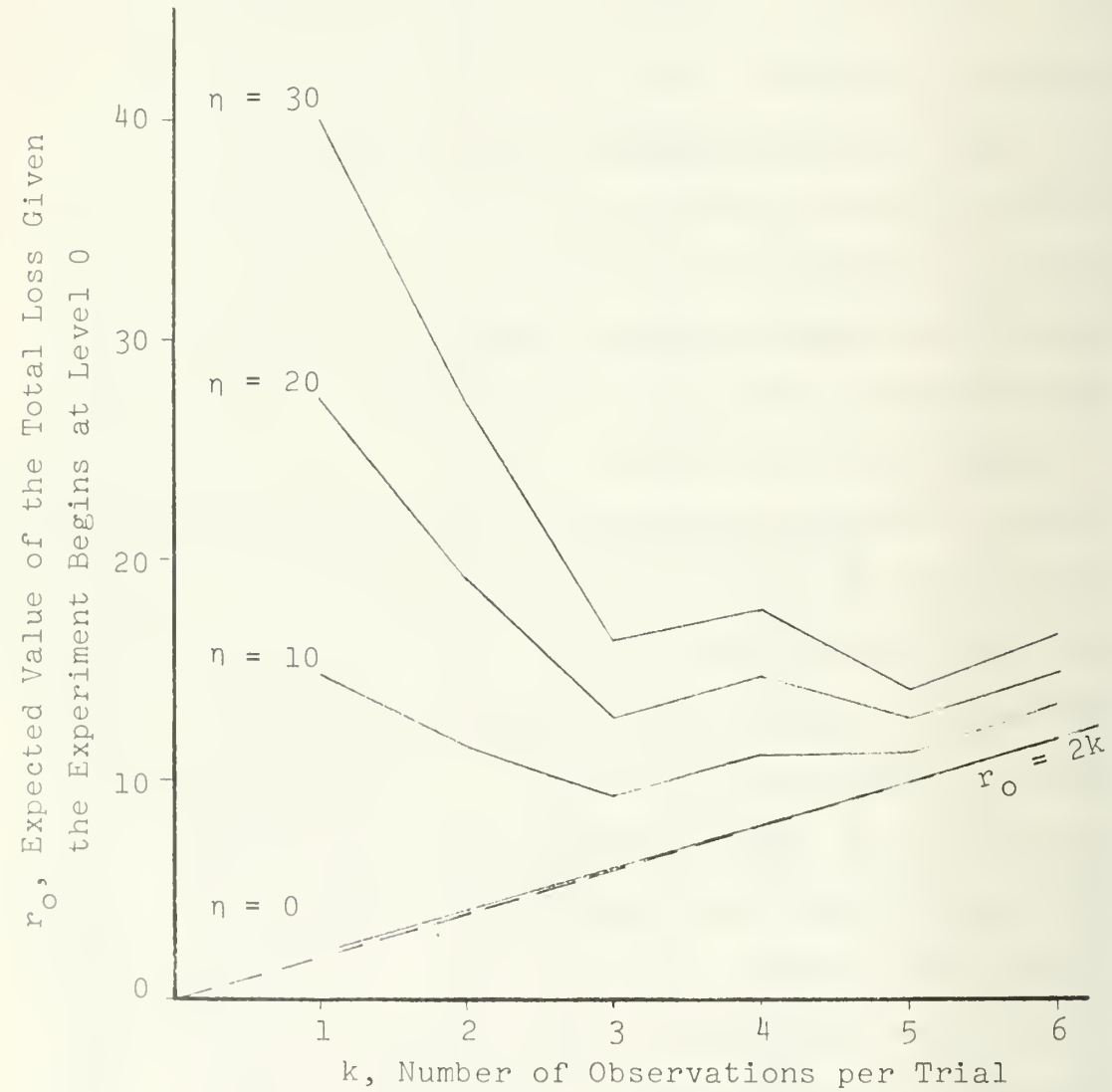

Figure 2. Expected Value of the Total Loss Given the Experiment Begins at Level 0 as a Function of the Number of Observations per Trial and the Magnitude of the Loss Parameter η for Procedures $W(\delta_1,k,0,1)$ and a Quadratic Loss Structure.

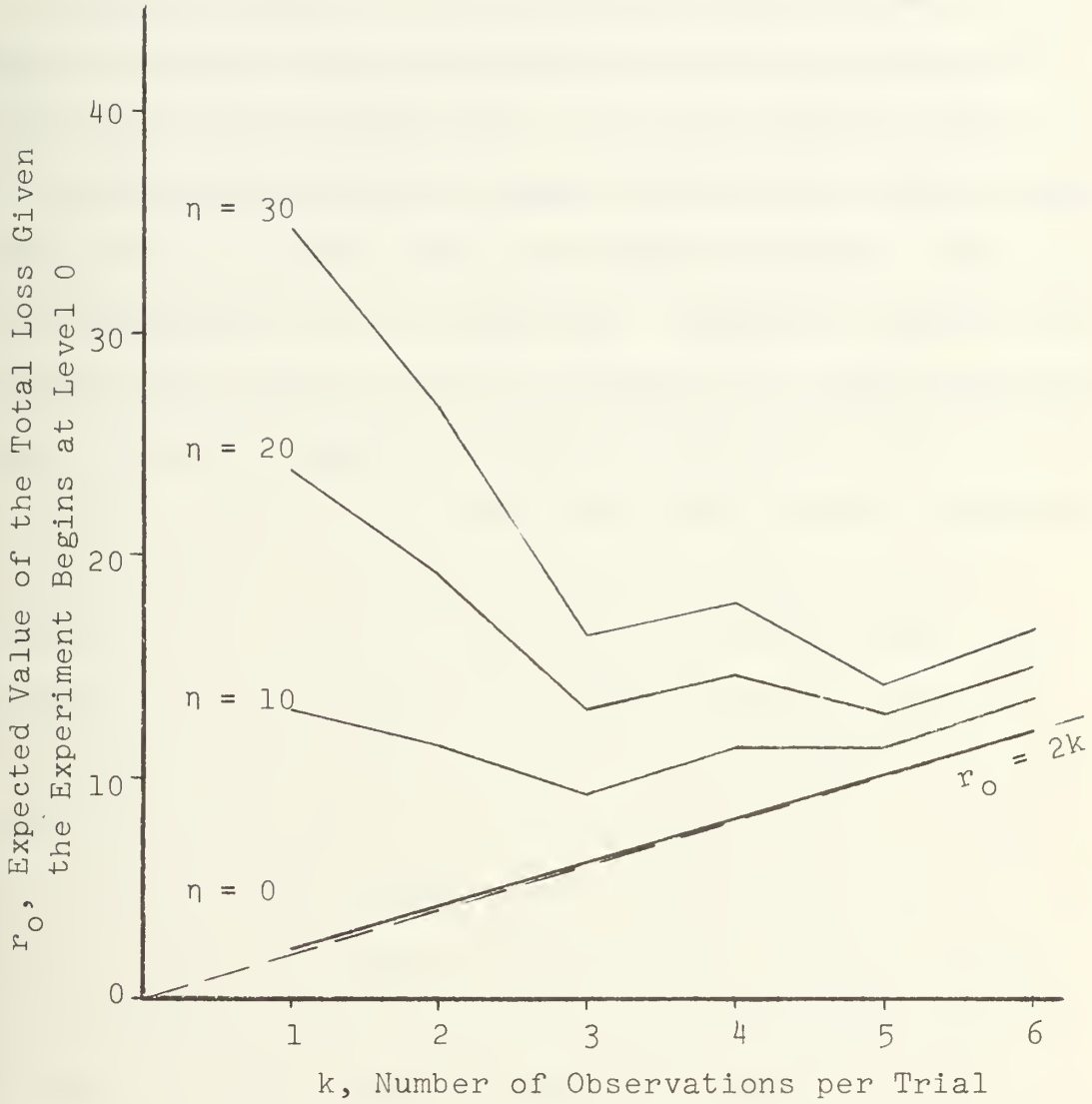

Figure 3. Expected Value of the Total Loss Given the Experiment Begins at Level 0 as a Function of the Number of Observations per Trial and the Magnitude of the Loss Parameter η for Procedures $W(\delta_1,k,0,1)$ and a Linear Loss Structure.

asymptotic to the line $r_0 = 2k$, an asymptote suggested by (67); and the slopes of the curves, in general, are decreasing for higher values of η.

The results from the evaluation of some procedures in Γ_1 are helpful in several important ways. First, they tend to validate the evaluation methodology presented earlier; that is, the results were consistent with expectations. Secondly, they give insight into the selection of one of the criteria discussed, and lastly, some consequences of the model have been brought out; for example, it might be that level 0 should be defined as the midpoint of a closed interval.

IV. APPLICATION TO THE ELEVATION PROCEDURE OF THE FIELD ARTILLERY PRECISION REGISTRATION

Field artillery weapons are normally positioned so that muzzle blasts from the weapons cannot be seen by ground observers of an opposing force. As a consequence, projectiles fired from the weapons must be fired at targets which usually are not visible from the weapon position. Fire delivered under these circumstances, normally from weapons having a relatively low muzzle velocity, is called indirect fire. Given the location of the weapon, the location of the target, and ballistic data based upon standard weapon-weather-ammunition conditions, one can compute firing data--direction (called deflection), elevation angle, and ammunition fuze setting--which under standard conditions will cause the mean point of impact of the projectiles to be on the target. In practice, standard conditons rarely exist and there are usually some location and delivery errors. Thus, if a number of projectiles, later referred to as rounds, is fired at a target using standard data and the mean point of impact is determined, there is some cummulative error due to nonstandard conditions which causes the mean point of impact to be away from the target. The precision registration technique is one method which is used by U.S. Army and Marine Corps field artillery units to determine the magnitude of a correction for this error.

A precision registration is performed to determine, by adjustment, the firing data which will cause the mean point of impact of a group of rounds to be at a point of known location, called the registration point [7]. Note that this is at least a two dimensional problem. An adjusted deflection and elevation must be determined and if an adjusted fuze setting is desired so that air bursts are achieved, then it becomes three dimensional. Since the discussion ultimately concerns only the adjusted elevation it is assumed in that which follows that an adjusted fuze setting is not desired. In a precision registration, firing is conducted in two phases using only one weapon. In the first phase, called the adjustment phase, the objective is to adjust the mean center of impact of a group of rounds to within a specified distance of the registration point. This is accomplished by use of an observer who determines corrections and transmits them to fire direction personnel. Normally the observer uses a bracketing procedure. In the second phase, called the fire-for-effect phase, a number of rounds is fired to permit refinement of the mean center of impact. Here the observer transmits quantal response data only; that is, he observes where the round bursts in relation to the registration point and transmits this information to fire direction personnel. The number of rounds used in the fire-for-effect phase varies randomly and the phase ends when a set of decision criteria are satisfied.

Note the similarity between the fire-for-effect phase of the precision registration and the bio-assay experiment. The first round fired in the fire-for-effect phase is the first trial of an "experiment" being performed to estimate the adjusted firing data. One cannot measure the set of "critical" firing data which causes the mean point of impact to be at the registration point. He can only select a set of data and then observe where the round bursts in relation to the registration point. In addition, it is well known that if a large number of bursts is observed, then the distribution of the bursts about the registration point is approximately normal. Also, the variance is known from previous test firings. Hence, the fire-for-effect phase of the registration may be viewed as an experiment[10] to estimate the mean of a bivariate normal distribution having known covariance matrix.

Separate procedures, implemented simultaneously, are currently used to determine the adjusted deflection and the adjusted elevation in the fire-for-effect phase. One is not dependent on the other in that the adjusted deflection may be determined first, or vice versa. It thus seems reasonable to assume that the fire-for-effect phase consists of two independent procedures. In that which follows this assumption has been made; that is, the elevation procedure

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[10]Thus, when "the experiment" is referred to it is clear that what is meant is "the fire-for-effect phase of the registration."

is viewed as an experimental procedure for estimating the mean of a univariate normal distribution. It is also assumed that firing is to be conducted at elevation angles less than 45 degrees, that there is independence between bursts, and that the registration point is, in fact, a point on the real line so that the probability of hitting the registration point is zero. The purpose of this chapter is to show that the elevation procedure currently in use belongs, under the assumptions made above, to the class of sequential procedures $\mathcal{C}$, defined in I. A second purpose is to compare the efficiency of the current procedure with some alternative procedures in $\mathcal{C}$.

A. THE CURRENT ELEVATION PROCEDURE

The current elevation procedure, later referred to as the "current procedure," will now be described within the framework of an experiment. The fire-for-effect phase is entered when it is determined that a correction given by the observer will move the burst within a specified distance of the registration point. The elevation determined from this, the observer's last correction, establishes the initial level of the experiment and the first trial is performed by firing one round at the registration point using the elevation corresponding to this initial level. If the first round bursts at a range greater (less) than the range to the registration point, it is said to be over (short) and the next level for the experiment is determined by

decreasing (increasing) the elevation by an amount which will move the mean center of impact four range probable errors.[11] Succeeding trials of the experiment are performed by firing one round at succeeding levels until a burst is observed to be short (over). This establishes a one fork elevation bracket; that is, an over and a short, fired at levels one fork apart, have been observed. The next level of the experiment is the midpoint of the one fork elevation bracket, and the next trial is performed by firing three rounds at this level. If the majority of the bursts are over (short), then the last level of the experiment is the elevation corresponding to the short (over) end of the one fork elevation bracket and the last trial of the experiment is performed by firing two rounds at this level.

In the experimental procedure just described note that: the number N, of rounds fired is a random variable since more than two trials may be necessary to establish a one fork elevation bracket; the number k, of rounds fired per trial is not constant; and successive trials of the experiment are not performed at successive levels. Hence, the current elevation procedure of the field artillery precision registration technique is a member of the class of procedures $\mathcal{C}$ defined in I and can be modeled by the general procedure $W(\delta, k_t, k_t^{o}, C_t)$ in $\mathcal{C}$.

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[11] One range probable error, PE_r, is approximately 0.67 standard deviations. Four PR_r is called a fork.

Let $\mathcal{M}' = \{\ldots, -1, 0, 1, \ldots\}$ be a set of equally spaced levels having interval m = 1/2 fork where the level 0 corresponds to the registration point. Let $\mathcal{M} = \{-3, \ldots, 0, \ldots 3\}$ be a subset of $\mathcal{M}'$. Then the current elevation procedure can be described as follows. The experiment (the fire-for-effect phase) begins at some (elevation) level Y_1 in $\mathcal{M}$ with one observation on the first trial. Here, Y_1 is in $\mathcal{M}$ because the adjustment phase does not end until the mean center of impact is adjusted to within a specified distance of the registration point. An observation is a response (an over) or a nonresponse (a short) depending upon whether the round bursts at a range greater than or less than the range to the registration point. If G is the distribution of bursts about the registration point when the mean point of impact is at the registration point, then the probability of a response at level j is G(j); j = -3,...,3. This is just the probability that the adjusted elevation is less than the elevation used to perform a trial of the experiment at level j. The experiment continues at levels $Y_2, \ldots, Y_{n^*}$, with $k_2, \ldots, k_{n^*}$ observations per trial, determined by:

$$Y_{t+1} = \begin{cases} Y_t + C_t \cdot m & \text{if } 0 \leq J_t \leq k_t^{o} \\ Y_t & \text{if } k_t^{o} < J_t < k_t - k_t^{o}, \quad t \geq 1 \\ Y_t - C_t \cdot m & \text{if } k_t - k_t^{o} \leq J_t \leq k_t \end{cases}$$

where J_t is the number of responses on the t-th trial and

$$C_t = \begin{cases} 2 & 1 \leq t \leq n^* - 3 \\ 1 & n^* - 2 \leq t \leq n^* - 1 \end{cases}$$

$$k_t^{\circ} = \begin{cases} 0 & 1 \leq t \leq n^* - 2 \\ 1 & t = n^* - 1 \end{cases}$$

$$k_t = \begin{cases} 1 & 1 \leq t \leq n^* - 2 \\ 3 & t = n^* - 1 \\ 2 & t = n^* \end{cases}$$

are sequences of integers. The rule δ in $W(\delta, k_t, k_t^{\circ}, C_t)$ is to stop the experiment two trials after the first reversal in the sequence of responses $\{J_t\}$ is obtained. For example, if $J_1 = 0$, $J_2 = 0$ and $J_3 = 1$, then two more trials would be performed and $n^* = 5$. Note that since $\mathcal{M} = \{-3, \ldots, 0, \ldots, 3\}$ as few as seven rounds and as many as nine rounds may be required for the experiment since if it starts at level -3, as many as four rounds may be required to establish a one fork elevation bracket.

When firing has been completed, the adjusted elevation is computed using the data from the last two trials and an earlier trial performed at the same level as the last trial. Thus six observations are used from trials performed at two of the levels in $\mathcal{M}'$. The adjusted elevation is the mean elevation fired to obtain this group of six observations $\bar{Y}$, plus an elevation change ΔY, determined by:

$$\Delta Y = - \frac{\text{difference in number of overs and shorts}}{2 \times 6 \text{ (the number of observations)}} \times \text{fork}, \tag{69}$$

where the sign of the right side is negative because a positive difference in the number of overs and shorts indicates the mean elevation fired is greater than the adjusted elevation. Let Δ be the difference in overs and shorts, then (69) can be written

$$\Delta Y = - \frac{\Delta \cdot m}{6}$$

where use has been made of the fact that $m = \frac{1}{2}$ fork. Let J_T be the total number of responses (overs), then

$$\Delta = J_T - (6 - J_T) = 2 J_T - 6$$

so that the adjusted elevation may be computed using the estimator

$$\hat{Y} = \frac{1}{2} \sum_{t=n^*-1}^{n^*} Y_t - \frac{m}{3} J_T + m \qquad (70)$$

Note that if the total number of responses is three (that is, if there are three overs and three shorts), then the last two terms of (70) sum to zero.

Given the triplet ($\mathcal{M}$, W, $\hat{Y}$) the procedure transition and iteration matrices can now be determined from (10) and (11). Let Q be a discrete random variable with distribution function F which denotes the level Y_1 in $\mathcal{M}$ at which the experiment begins. Q is a random variable because Y_1 is determined by a process of adjustment which is subject to random error. Indeed, if the interval m between levels in $\mathcal{M}$ is large, then the distribution may concentrate nearly

all its mass at level 0. Let M be a discrete random variable which denotes the level in $\mathcal{m}$ which represents the adjusted elevation. Recall from II, that $\mathcal{m}$ represents a finite set of midpoints of disjoint intervals, the union of which forms an open interval on the real line. Thus the adjusted elevation can be represented by one of the levels in $\mathcal{m}$ and the joint conditional probability $f_{M,N|Q}(j, n|i)$, $(i,j = -3,\ldots 0,\ldots,3;\ n = 7,8,9)$ that the experiment ends at some level j in $\mathcal{m}$ after n rounds have been fired given it begins at level i, can be found using (7), (8) and a logic tree similar to that shown in Figure 1. Let $\bar{q}$, a (7 x 1) column vector of probabilities that the experiment starts at level i, be a prior distribution on Q. The i,j-th element of the Procedure Transition Matrix A (the probability that the registration ends at level j given the fire-for-effect phase begins at level i) is

$$a_{ij} = \sum_{n=7}^{9} f_{M,N|Q}(j, n|i) \qquad (71)$$

and the probability that the registration ends at level j, given by (35), is

$$\Pr[M = j] = g_M(j) = \bar{q}' A \ .$$

Likewise, the i,j-th element of the Procedure Iteration Matrix B (the probability that the registration ends after j rounds have been fired, given the fire-for-effect phase begins at level i) is

$$b_{ij} = \sum_{m=-3}^{3} f_{M,N|Q}(m, j|i) \qquad (72)$$

and the probability that the registration requires j rounds, given by (37) is

$$\Pr[N = j] = g_N(j) = \bar{q}' B.$$

The current procedure can now be evaluated using a loss plus cost objective function in a manner analogous to Wald's approach. If H is a random variable which denotes the loss associated with a registration that ends at some level in $\mathcal{M}$ other than level 0 and the loss is measured in units of the cost of a round, then the total loss L, also measured in units of the cost of a round, is given by (33). That is,

$$L = H + N.$$

Here, other costs incidental to the experiment are assumed to be constant and thus may be neglected for purposes of this evaluation. It appears that (33) is a reasonable objective function for the elevation procedure of the precision registration. An adjusted elevation which makes the total loss as low as possible would be preferred. Consider, for example, the consequences of deciding that the adjusted elevation is at level -3. This means that subsequent firing using the adjusted elevation as a basis for corrections will, on the average, be short by six range

probable errors, a situation that may be intolerable if fires are to be delivered close to friendly forces. Indeed, it is reasonable that, in certain circumstances, a high loss be associated with making such a decision. Although the nature of the loss function h(j) may not be known, one of several intuitively plausible loss functions may be used for the evaluation. This point will be considered below in more detail.

If $\bar{h}_1$ is a (7 x 1) column vector whose j-th component is the loss associated with a registration which ends at level j and $\bar{n}_1$ is a (3 x 1) column vector, $\bar{n}_1 = (7, 8, 9)'$, then the expected total loss, given by (39), is

$$E[L] = \bar{q}'(A\,\bar{h}_1 + B\,\bar{n}_1)$$

where use is made of (35) and (37). Likewise if $\bar{h}_2$ is a (7 x 1) column vector whose j-th component is the square of the loss associated with a registration which ends at level j and $\bar{n}_2$ is a (3 x 1) column vector $\bar{n}_2 = (49, 64, 81)'$, then the variance of the total loss, given by (56), is

$$V[L] = \bar{q}'\,\bar{c} - \bar{q}'\bar{D}\,\bar{q}$$

where $\bar{c} = A\,\bar{h}_2 + B\,\bar{n}_2 + 2C\,\bar{h}_1$ is a (7 x 1) column vector, $D = (A\,\bar{h}_1 + B\,\bar{n}_1)(A\,\bar{h}_1 + B\,\bar{n}_1)'$ is a (7 x 7) positive semi-definite symmetric matrix having rank one, and $C\,\bar{h}_1$ is a (7 x 1) column vector whose i-th component is the product moment of the random variables H and N given the experiment

starts at level i. The reader is referred to III for a more detailed derivation of the variance of the total loss.

Given $\mathcal{M}$, the statistical properties of the current procedure can be determined using (71) and (72). For example, the probability that the procedure uses j rounds, given fire-for-effect starts at level i is given by (72). The efficiency of the procedure can be determined for any finite loss structure from (39) and (56). Further discussion of the current procedure will be deferred to a later paragraph where its properties and efficiency can be compared with those of two alternative procedures which are discussed in the next section.

B. ALTERNATIVE ELEVATION PROCEDURES

An elevation procedure which was used in the precision registration technique prior to 1965 will now be discussed. For convenience, this procedure will be referred to as the "old procedure." Essentially, the old procedure differs from the current procedure in that a one fork elevation bracket is not obtained prior to firing the initial group of three rounds and all subsequent firing is in groups of three rounds. That is, the first trial of an experiment performed with the old procedure begins at some level Y_1 in $\mathcal{M}$ with three observations. The experiment continues at levels $Y_2,\ldots,Y_{n^*}$, with three observations per trial, determined by:

$$Y_{t+1} = \begin{cases} Y_t + m & \text{if } J_t = 0 \\ Y_t & \text{if } J_t = 1,2, \\ Y_t - m & \text{if } J_t = 3 \end{cases} \qquad t \geq 1$$

where m = 1/2 fork and J_t is the number of responses (overs) on the t-th trial. The experiment ends when both responses and nonresponses have been obtained on two successive trials. Note that the old procedure can be denoted $W(\delta_1, 3, 0, 1)$, a procedure in Γ_1. Note also that $6 \leq n \leq 21$ since the experiment could conceivably start at level -3 and not obtain a response until level 3 is reached. When firing is completed the estimator given by (70), where

$$J_T = \sum_{t=n^*-1}^{n^*} J_t,$$

is used to compute the adjusted elevation.

A second alternative to the current procedure is the modified up-and-down method with a "nominal" sample size of six described by Dixon [4]. This procedure will be referred to as the "bio-assay procedure." Under this procedure, the experiment begins at some level Y_1 in $\mathcal{M}$ with one observation on the first trial. The experiment continues at levels $Y_2,\ldots,Y_{n^*}$, with one observation per trial, determined by:

$$Y_{t+1} = \begin{cases} Y_t + m & \text{if } J_t = 0 \\ Y_t - m & \text{if } J_t = 1 \end{cases}, \qquad t \geq 1$$

where m = 1/2 fork and J_t is the number of responses (overs) on the t-th trial. From II this procedure can be denoted $W(\delta_2, 1, 0, 1)$ where the rule δ_2 is to perform four more trials after the first reversal in the sequence of responses $\{J_t\}$. Here $6 \leq n \leq 11$ because the first reversal could occur as soon as the second trial and as late as the seventh trial (that is, start at level -3) and not obtain a response until level 3 or vice versa). Note that the rule δ_2 is more liberal than δ_1 in that its use permits four more trials (instead of no more) after the first reversal. When firing is completed, the adjusted elevation is determined from the estimator

$$\hat{Y} = Y_{n*} + \kappa m \tag{73}$$

where Y_{n*} is the last elevation fired, κ is a constant (see Table I of reference 4) which depends upon the sequence $\{J_t\}$ and m is the interval between levels. Equation 73 was empirically constructed by Dixon and is essentially equivalent to

$$\hat{Y} = \frac{1}{6} \sum_{t=n*-4}^{n*+1} Y_t,$$

for a nominal sample size of six. Here, Y_{n*+1} is the level that would be used if one more trial was conducted. For example if m = 1, $\{Y_t\} = \{0,1,2,1,2,1,0\}$, and $\{J_t\} = \{0,0,1,0,1,1,1\}$, then $Y_{n*+1} = -1$ and $\hat{Y} = \frac{5}{6}$ while use of (73) gives $\hat{Y} = 0.732$.

C. AN EVALUATION OF THE CURRENT ELEVATION PROCEDURE AND SOME ALTERNATIVES

A comparison of the three alternative elevation procedures described above is presented in this section. For convenience the alternatives will be referred to as follows:

A - The old procedure

B - The bio-assay procedure

C - The current procedure

The objective function defined by (33) was used with two loss functions given by

$$h_1(j) = \eta j^2 \qquad j = -3,\ldots,0,\ldots,3 \tag{74}$$

$$h_2(j) = \begin{cases} \eta j^2 & j = -3, -2, -1 \\ 0 & \text{otherwise} \end{cases} \tag{75}$$

where η is an arbitrary constant which determines the magnitude of the loss. Note that h_1 is both quadratic and symmetric in j, while h_2 is asymmetric. The loss function h_2, which appears to be a reasonable when fires are to be delivered just beyond friendly forces, associates a high loss with the decision that the adjusted elevation is less than the true adjusted elevation and no loss with the converse decision. The criteria used in the comparisons are given by (43), (45), (58), and (60).

A computer program, similar to that shown on page 127, was written in FORTRAN IV and ran on an IBM 360 computer to

evaluate equations 39 and 56 for the loss functions h_1 and h_2 and various values of the loss parameter. The set of all unit vectors in Euclidean seven space, $\bar{e}_i$ in E^7, was taken as a set of prior distributions on Q.

A summary of the statistical properties of the various procedures given that the fire-for-effect phase of the registration begins at level 0 is given in Table III. The data in the table are the center rows of A and B matrices which are shown on pages 118, 121, and 124, respectively for procedures A, B, and C. The table is included to permit an appreciation of the relative merit of the various procedures given the fire-for-effect phase starts at level 0. Data for other starting levels is contained in the other rows of the A and B matrices. The data contained in the table should not be used as a basis of deciding that a particular procedure is best; rather, one of the criteria discussed III A 1, III A 2, III B 1, or III B 2 might be used.

Computations of the expected loss and variance of the loss are shown for both loss functions and seven values of the loss parameter beginning on pages 119, 122, and 125 for procedures A, B, and C, respectively. The first column in each matrix, the expected loss and variance of the loss when the loss parameter η is zero, is equivalent to the expected value and variance of the number N, of rounds fired during the fire-for-effect phase. To see this, the

TABLE III

STATISTICAL PROPERTIES FOR PROCEDURES A, B AND C GIVEN FIRE-FOR-EFFECT BEGINS WITHIN ONE PE_r (LEVEL 0) OF THE REGISTRATION POINT

Probability of Ending at Level M Given a Start at Level 0							
Procedure	-3	-2	-1	0	1	2	3
A	Nil	0.00009	0.17179	0.65265	0.17179	0.00009	Nil
B	Nil	0.00046	0.18637	0.62635	0.18637	0.00046	Nil
C	Nil	0.00217	0.18747	0.62073	0.18747	0.00217	Nil

Probability of Firing N Rounds Given a Start at Level 0					
Procedure	6	7	8	9	10
A	0.99983	0	0	0.00017	0
B	0.91134	0.08835	0.00031	0	0
C	0	0.99651	0.00349	0	0

reader is referred to the discussion of the expected value and variance of the loss in III. Figure 4 shows plots of the expected value and variance of the total number of rounds as a function of the starting level. There, the indicated Bayes procedure relative to $\bar{e}_i \in E^7$ are: Procedure A for i = 0, ±1, Procedure B for i = ±2, and Procedure C for i = ±3. And the minimax procedure is C. Thus, if there is no loss associated with deciding that the adjusted elevation is at level M, it is seen that use of the current procedure gives protection against high losses resulting from staring away from level 0. This result is characteristic of a minimax procedure. Notice that variance criteria provide different candidates for the optimal procedure. Procedure A is the minimum variance procedure relative to $\bar{e}_0 \in E^7$ while Procedure C is both the minimum variance procedure relative to $\bar{e}_1 \in E^7$ for $i \neq 0$ and the minimax procedure. These results are interpreted to mean that, in the long run, the use of Procedure A or B may result in a smaller expenditure of ammunition; but, the use of Procedure C will result in a near constant expenditure of ammunition per registration. It is interesting to note the differences between Procedures A and C. Note from Figure 4 that use of Procedure A results in a smaller expected number of rounds if the fire-for-effect phase starts at level i, i = 0, ±1. In practice, however, the starting level cannot be fixed with certainty. In fact,

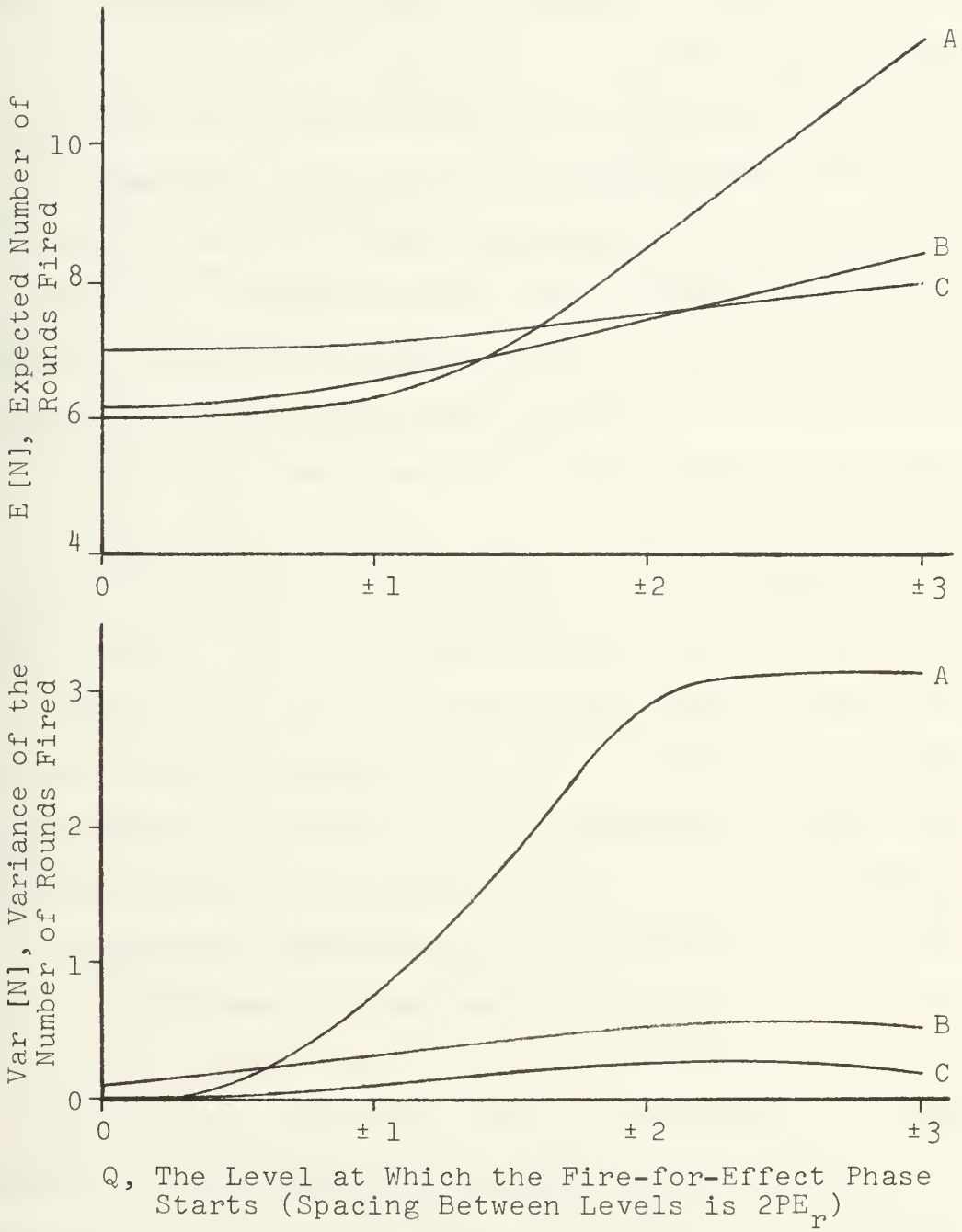

Figure 4. Expected Value and Variance of the Number of Rounds Fired in Fire-for-Effect for Procedures A, B and C as a Function of Starting Level (Here the Loss Parameter η is Zero).

the basis for adoption of the current procedure (C), was that an occasional observer error in the adjustment phase would, when using the old procedure (A), result in an excessive number of rounds [16]. While the results shown in Figure 4 substantiate this decision, the reader is reminded that here the loss parameter is zero; that is, to choose a procedure on the basis of expected value and variance in the number of rounds may be equivalent to deciding that there is no risk associated with deciding that the adjusted elevation takes on some value other than the true adjusted elevation.

Suppose the loss structure can be represented by h_1, the quadratic loss function given by (74). Figure 5 shows the expected value and variance of the total loss as a function of starting level for a quadratic loss structure when the loss parameter is 50. Note that Procedures A and/or B satisfy all of the criteria. Procedure A is both the Bayes and minimum variance procedure relative to $\bar{e}_i \in E^7$ for $i = 0, \pm 1$. Procedure B is the Bayes and minimum variance procedure relative to $\bar{e}_i \in E^7$ for $i = \pm 2, \pm 3$. In addition, Procedure B is both the minimax and minimax variance procedure, replacing Procedure C which satisfied these criteria under the no loss assumption. In fact, it appears that Procedure B gets "better" as the distance between the starting level and the registration point increases. This result is surprising and may be due to the

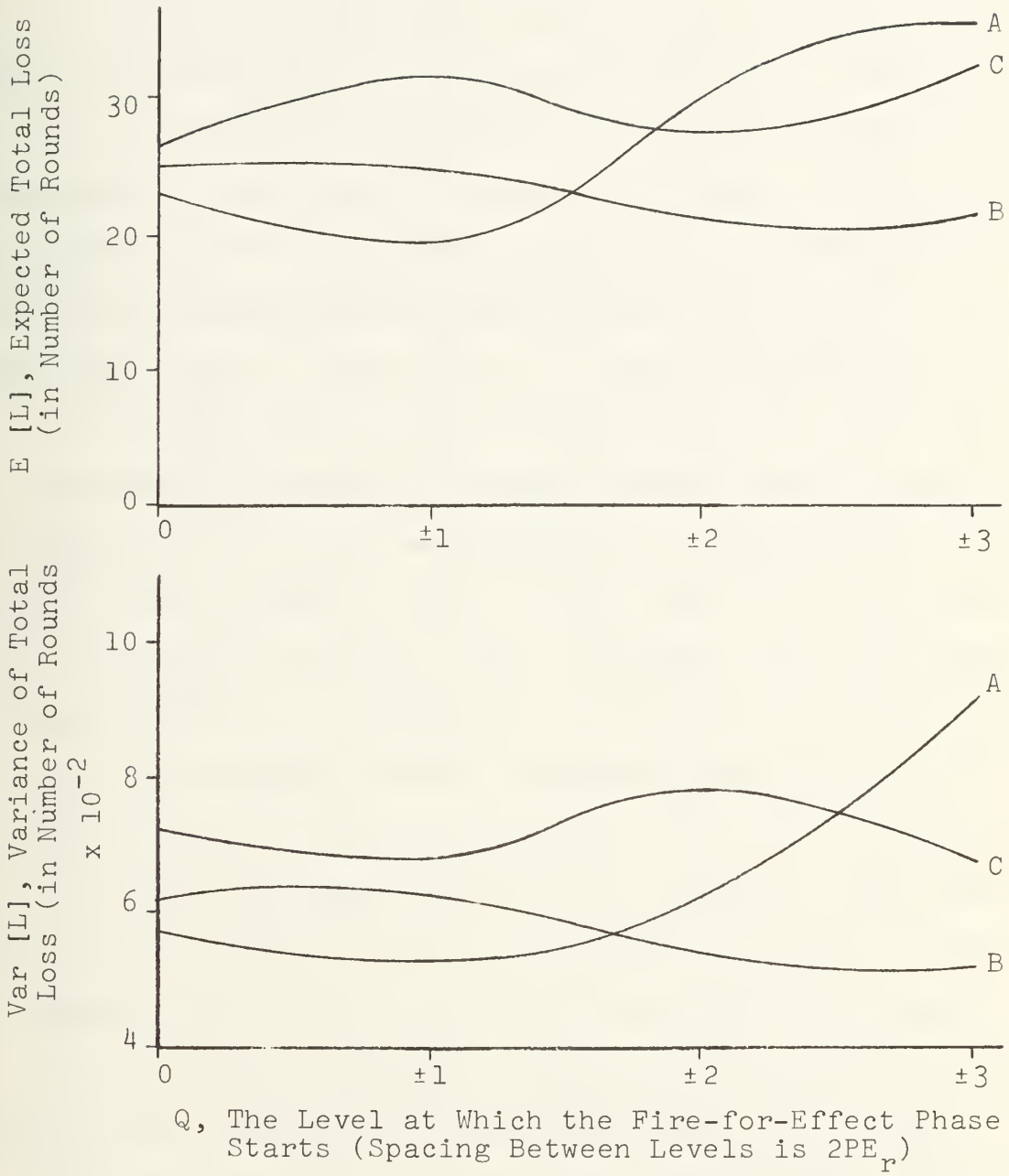

Figure 5. Expected Value and Variance of the Total Loss for Procedures A, B and C as a Function of the Starting Level When the Loss Structure is Quadratic and the Loss Parameter η is 50.

use of (73) in computing the adjusted elevation. Note also the magnitude of the variance of the total loss. Table IV shows optimal procedures under the various criteria for the values of the loss parameter η considered when the loss structure is quadratic. It can be seen that for $\eta > 10$, the ordering among the alternative procedures is independent of the magnitude of the loss parameter η. Recall this was not true in III C 2 where evaluation of some procedures in Γ_1, including Procedure A, was discussed. Here, this result essentially means that one need not be concerned with determining how large a loss may be suffered; rather he should deal with the more important issue of determining whether or not a loss will in fact be sustained and if so, whether or not a quadratic loss structure is appropriate. It should be noted that the result may not hold for higher values of the loss parameter. Another interpretation of the data in Table IV is that Procedure B is more conservative than Procedure A. This can be seen by noting first that Procedure B is, for $\eta > 0$, both a minimax and minimax variance procedure, and that Procedure B is both a Bayes and minimum variance procedure relative to $\bar{e}_i \in E^7$ for $i = \pm 2, \pm 3$. In other words, use of Procedure B gives protection against high losses and variances resulting from starting the fire-for-effect phase away from level 0. Note however that in case it can be assured that the fire-for-effect phase starts at levels 0 or ± 1 (that is, within 3 PE_r

TABLE IV

OPTIMAL PROCEDURES FOR A QUADRATIC LOSS STRUCTURE

Criterion	Value of Loss Parameter, η					
	0	10	20	30	40	50
Bayes	A(0,±1) B(±2) C(±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)
Minimax	C	B	B	B	B	B
Minimum Variance	A(0) C(±1,±2,±3)	A(0) B(±1,±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)	A(0,±1) B(±2,±3)
Minimax Variance	C	B	B	B	B	B

NOTE: The notation A(0,±1) means that procedure A is optimal relative to the prior distributions $\bar{e}_i \in E^7$, for i = 0,±1.

of the registration point since level 1 is the midpoint of an interval of length 2 PE_r) then Procedure A is best for $\eta > 10$.

Perhaps a more realistic loss structure in some cases is one which is asymmetric. If it can be represented by h_2, given by (75), then the decision maker is confronted with a different situation. Recall the interpretation of h_2 is that there is a high loss associated with deciding the adjusted elevation is less than the true adjusted elevation and no loss is associated with deciding that it is greater than the true adjusted elevation. Figure 6 shows the expected value and variance of the total loss as a function of starting level for an asymmetric loss structure when the value of the loss parameter is 50. Note the trends in this situation. Procedure A varies widely depending upon the starting level. Procedure B again appears to be better when the fire-for-effect phase starts away from the registration point, particularly from <u>below</u>. Procedure C is fairly consistent at about midway between the extremes noted for Procedure A. Here also, Procedures A and/or B dominate Procedure C with respect to all the criteria. Note that both the expected value and variance of the total loss take on larger values than they did in the previous case. Table V shows the procedures which are optimal under the various criteria for the values of η considered when the loss structure is asymmetric. Note

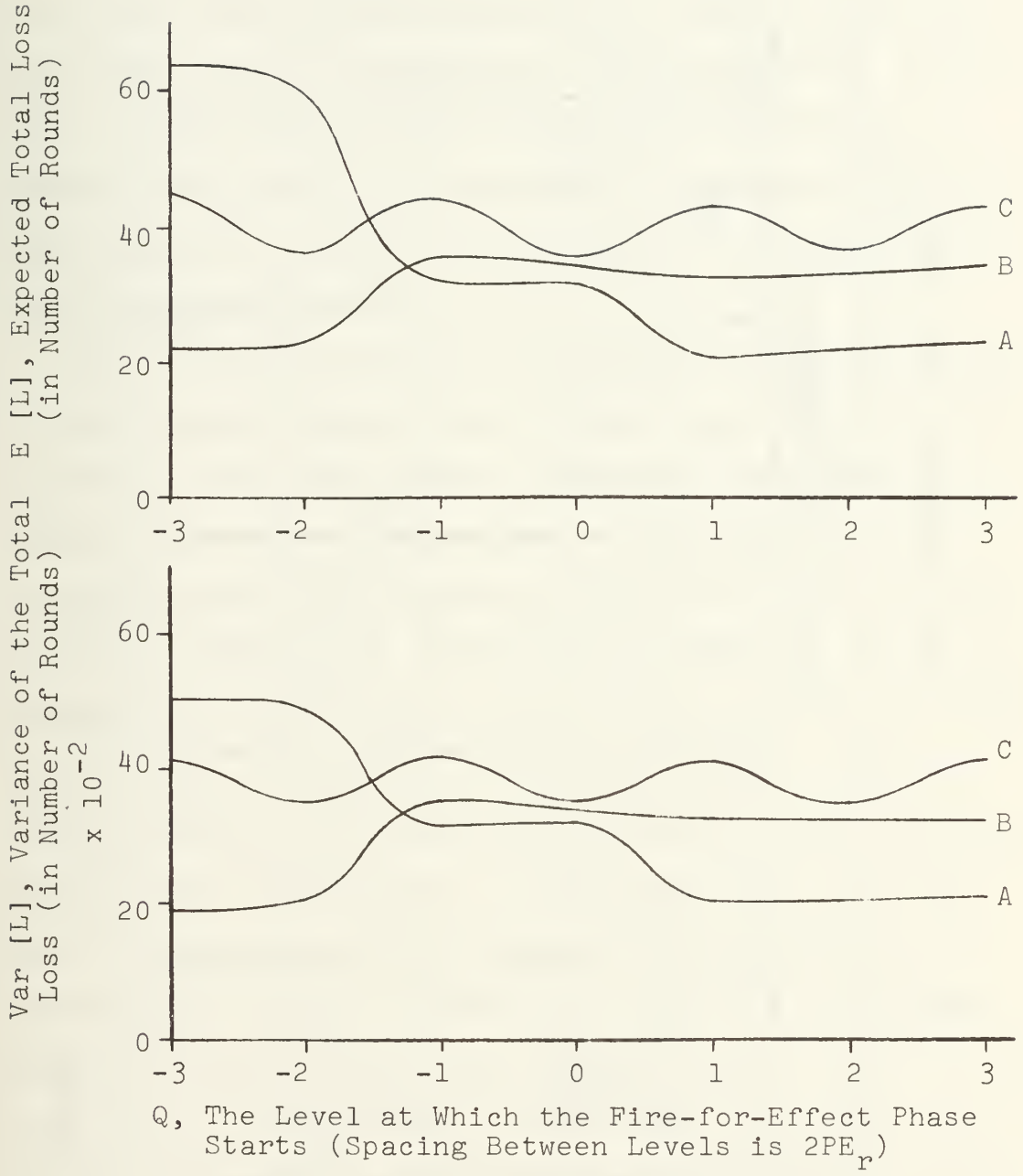

Figure 6. Expected Value and Variance of the Total Loss for Procedures A, B and C as a Function of Starting Level When the Loss Structure is Asymmetric and the Loss Parameter η is 50.

TABLE V

OPTIMAL PROCEDURES FOR AN ASYMMETRIC LOSS STRUCTURE

Criterion	Value of Loss Parameter, η					
	0	10	20	30	40	50
Bayes	A(0,±1) B(±2) C(±3)	A(0,±1,2) B(-2,±3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)
Minimax	C	B	B	B	B	B
Minimum Variance	A(0) C(±1,±2,±3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)	A(0,±1,2,3) B(-2,-3)
Minimax Variance	C	B	B	B	B	B

NOTE: The notation A(0,±1) means that procedure A is optimal relative to the prior distributions $\bar{e}_i \in E^7$, for i = 0, ±1.

that in this case the ordering is also unchanged for $\eta > 10$. Again it is seen from the data in Table V that Procedure B tends to be conservative. It is interesting to observe that if an asymmetric loss structure is valid, then the expected value and variance of the loss can be greatly reduced by use of Procedure A and by assuring that the fire-for-effect phase always begins at a point beyond the registration point.

In concluding, it should be emphasized that the purpose of this chapter has not been to select an optimal elevation procedure, rather to show how the current procedure can be modeled and to define its degree of optimality by means of comparison with some alternative procedures. The current procedure was shown to be conservative under a no loss assumption; that is, it was found to be a minimax procedure when the basis of the evaluation and comparison was the expected number of rounds fired during the fire-for-effect phase of the registration. It was shown that if a loss structure exists and can be represented by either of the loss functions h_1 or h_2 for $0 < \eta \leq 50$, then the current procedure is not optimal with respect to any of the criteria considered. In particular, if the loss structures considered are valid, then for $\eta > 0$ the bio-assay procedure strictly dominates the current procedure and the old procedure dominates the current procedure relative to prior distributions $\bar{e}_i \in E^7$ for $i = 0, \pm 1$. These results

suggest that further study of the elevation procedure is warranted. Indeed, it may be possible, using the bio-assay procedure, to enter fire-for-effect more rapidly than is currently the case.

V. SUMMARY AND CONCLUSIONS

A class of sequential procedures for estimating the mean of a normal distribution function with known variance from quantal response data has been discussed. An experiment conducted using a procedure belonging to the class is characterized by: 1) a random number of trials, 2) a number of observations per trial that is not necessarily constant throughout the experiment, and 3) a sequence of trials that are not necessarily performed successively at equally spaced levels. It was seen that the general procedure $W(\delta, k_t, k_t^{o}, C_t)$, called a modified Tsutakawa random walk design, depends upon the stopping rule δ and the sequences $\{k_t\}$, $\{k_t^{o}\}$, and $\{C_t\}$. The class of general procedures was seen to include the up-and-down method and the Tsutakawa random walk design as special cases. Two matrices, the Procedure Transition Matrix and the Procedure Iteration Matrix, were found to characterize the general procedure. A technique of evaluating the efficiency of the general procedure was presented which is essentially an application of Wald's decision theory. It was seen that a loss plus cost objective function can be used along with one of the four decision criteria discussed to determine which among a set of alternative procedures is best. Matrix equations for the expected value and variance of the total loss were derived which depend upon the a priori

distribution on the starting level, the stopping rule, the estimator used to determine the mean, and the loss structure.

In the special case, procedures of the form $W(\delta_1, k, 0, 1)$ were examined in some detail for $k = 1,\ldots,6$ and a particular estimator. It was found that choosing a number k^* of observations per trial which minimizes the expected total loss is equivalent to choosing a Bayes procedure. It was also found that whether or not $k^* > 1$ depends upon the magnitude of the loss which is associated with the decision that the estimated mean has some value other than the true mean. In the special case, it was seen that the choice of a stopping rule is critical when defining a set of alternative procedures. A rule that stops the experiment too quickly may result in comparing experiments which have different sample sizes.

Three alternative elevation procedures for the field artillery precision registration technique, used by the U.S. Army and Marine Corps were discussed and compared. It was seen that the current procedure may be considered optimal when there is no loss associated with the decision that the adjusted elevation has some value other than the true adjusted elevation. When there is a loss associated with this decision which can be represented by one of the loss functions considered, it was seen that Dixon's modified up-and-down method strictly dominates the current

procedure. The author believes that this result is significant and that the up-and-down method should be further evaluated as a candidate elevation procedure for the precision registration technique.

It appears that further work in this general area of research would be justified. In particular, it would be interesting to study the various procedures under different stopping rules and to investigate the effect of the use of various estimators and loss structures. It would also be of interest to investigate multivariate analogs of some of the procedures. Indeed, it may be possible to model the fire-for-effect phase of a precision registration with a two or three dimensional version of the up-and-down method.

RESULTS FOR K = 1

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.08866	0.45567	0.0	0.41527	0.04040
-1	0.08866	0.45567	0.0	0.41527	0.04040
0	0.04433	0.45567	0.0	0.45567	0.04433
+1	0.04040	0.41527	0.0	0.45567	0.08866
+2	0.04040	0.41527	0.0	0.45567	0.08866

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.08866	0.45567	0.41527	0.04040
-1	0.54433	0.41527	0.04040	0.0
0	0.91134	0.08866	0.0	0.0
+1	0.54433	0.41527	0.04040	0.0
+2	0.08866	0.45567	0.41527	0.04040

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	3.40741	14.69800	25.98857	37.27914
-1	2.49607	13.78666	25.07724	36.36781
0	2.08866	12.97526	23.86183	34.74843
+1	2.49607	13.78666	25.07724	36.36784
+2	3.40741	14.69800	25.98857	37.27917

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	0.49956	10.53110	43.04419	98.03931
-1	0.33079	11.90668	45.96362	102.50293
0	0.08080	9.77675	35.63330	77.64966
+1	0.33079	11.90669	45.96362	102.50073
+2	0.49956	10.53110	43.04419	98.03687

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	3.40741	17.27919	31.15096	45.02272
-1	2.49607	16.36784	30.23962	44.11139
0	2.08866	14.74846	27.40823	40.06802
+1	2.49607	15.36784	30.23962	44.11140
+2	3.40741	17.27919	31.15096	45.02274

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	0.49956	98.03589	397.89966	900.09058
-1	0.33079	102.50049	406.99609	913.82007
0	0.08080	77.64812	300.65479	669.10132
+1	0.33079	102.50049	406.99609	913.81860
+2	0.49956	98.03589	397.89966	900.08911

RESULTS FOR K = 2

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.00786	0.36923	0.41527	0.20600	0.00163
-1	0.00913	0.36796	0.41527	0.20600	0.00163
0	0.00197	0.37303	0.25000	0.37303	0.00197
+1	0.00163	0.20600	0.41527	0.36796	0.00913
+2	0.00163	0.20600	0.41527	0.36923	0.00786

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.16946	0.62291	0.20600	0.00163
-1	0.79236	0.20600	0.00163	0.0
0	0.99607	0.00393	0.0	0.0
+1	0.79236	0.20600	0.00163	0.0
+2	0.16946	0.62291	0.20600	0.00163

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	6.07961	12.02184	17.96405	23.90625
-1	4.41853	10.37346	16.32837	22.28328
0	4.00786	11.54716	19.08646	26.62575
+1	4.41853	10.37345	16.32837	22.28326
+2	6.07961	12.02184	17.96405	23.90627

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	1.52164	27.99461	106.48950	237.00635
-1	0.67499	30.43230	112.67139	247.39160
0	0.01567	19.54962	77.75928	174.64551
+1	0.67499	30.43231	112.67139	247.39258
+2	1.52164	27.99461	106.48950	237.00610

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	6.07961	12.21169	18.34377	24.47583
-1	4.41853	10.58871	16.75890	22.92905
0	4.00786	11.62576	19.24367	26.86157
+1	4.41853	10.58871	16.75890	22.92905
+2	6.07961	12.21169	18.34377	24.47583

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	1.52164	36.69557	142.08862	317.70215
-1	0.67499	40.81877	154.05493	340.38477
0	0.01567	23.38771	92.48535	207.30908
+1	0.67499	40.81877	154.05493	340.38477
+2	1.52164	36.69563	142.08887	317.70215

RESULTS FOR K = 3

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.00070	0.33701	0.56768	0.09455	0.00007
-1	0.00071	0.16979	0.73488	0.09455	0.00007
0	0.00009	0.17179	0.65625	0.17179	0.00009
+1	0.00007	0.09455	0.73488	0.16979	0.00071
+2	0.00007	0.09455	0.56768	0.33701	0.00070

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.24310	0.66229	0.09455	0.00007
-1	0.90539	0.09455	0.00007	0.0
0	0.99983	0.00017	0.0	0.0
+1	0.90539	0.09455	0.00007	0.0
+2	0.24310	0.66229	0.09455	0.00007

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	8.55474	12.88558	17.21642	21.54726
-1	6.28403	8.94296	11.60189	14.26082
0	6.00052	9.43976	12.87900	16.31824
+1	6.28403	8.94296	11.60189	14.26083
+2	8.55474	12.88558	17.21642	21.54726

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	2.84300	22.46559	91.49805	209.93921
-1	0.77266	24.62582	87.82895	190.38185
0	0.00159	22.61777	90.43163	203.44360
+1	0.77266	24.62585	87.82895	190.38181
+2	2.84300	22.46561	91.49805	209.93921

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	8.55474	12.90084	17.24693	21.59303
-1	6.28403	8.95852	11.63301	14.30749
0	5.00052	9.44324	12.88597	16.32869
+1	6.28403	8.95852	11.63301	14.30750
+2	8.55474	12.90084	17.24693	21.59303

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	2.84300	23.19444	94.52173	216.82373
-1	0.77266	25.48325	91.24463	198.05663
0	0.00159	22.82376	91.21396	205.17212
+1	0.77266	25.48325	91.24464	198.05658
+2	2.84300	23.19444	94.52173	216.82373

RESULTS FOR K = 4

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.00006	0.35325	0.60357	0.04311	0.00000
-1	0.00007	0.16808	0.78873	0.04311	0.00000
0	0.00000	0.16406	0.67188	0.16406	0.00000
+1	0.00000	0.04311	0.78873	0.16808	0.00007
+2	0.00000	0.04311	0.60357	0.35325	0.00006

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.31020	0.64669	0.04311	0.00000
-1	0.95689	0.04311	0.00000	0.0
0	0.99999	0.00001	0.0	0.0
+1	0.95689	0.04311	0.00000	0.0
+2	0.31020	0.64669	0.04311	0.00000

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	10.93164	14.89656	18.86147	22.82639
-1	8.17245	10.28583	12.39922	14.51261
0	8.00003	11.28135	14.56268	17.84399
+1	8.17245	10.29584	12.39922	14.51261
+2	10.93164	14.89656	18.86147	22.82639

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	4.51190	15.55380	74.47876	181.28613
-1	0.66029	20.06314	72.83037	158.96165
0	0.00017	22.04895	88.19308	198.43286
+1	0.66029	20.06316	72.83032	158.96159
+2	4.51190	15.55379	74.47876	181.28613

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	10.93164	14.89785	18.86404	22.83025
-1	8.17245	10.28730	12.40216	14.51702
0	8.00003	11.28151	14.56299	17.84447
+1	8.17245	10.28731	12.40216	14.51702
+2	10.93164	14.89785	18.86404	22.83025

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	4.51190	15.61464	74.73462	181.87158
-1	0.66029	20.14548	73.15894	159.70058
0	0.00017	22.05843	88.22852	198.51050
+1	0.66029	20.14548	73.15889	159.70052
+2	4.51190	15.61462	74.73462	181.87158

RESULTS FOR K = 5

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.00001	0.39100	0.58935	0.01964	0.00000
-1	0.00001	0.06786	0.91249	0.01964	0.00000
0	0.00000	0.07031	0.85938	0.07031	0.00000
+1	0.00000	0.01964	0.91249	0.06786	0.00001
+2	0.00000	0.01964	0.58935	0.39100	0.00001

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.37136	0.60900	0.01964	0.00000
-1	0.98035	0.01964	0.00000	0.0
0	1.00000	0.00000	0.0	0.0
+1	0.98035	0.01964	0.00000	0.0
+2	0.37136	0.60900	0.01964	0.00000

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	13.24142	17.34796	21.45451	25.56107
-1	10.09821	10.97338	11.84855	12.72372
0	10.00000	11.40625	12.81251	14.21876
+1	10.09821	10.97338	11.84855	12.72372
+2	13.24142	17.34796	21.45451	25.56107

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	6.68277	10.15723	62.03687	162.32251
-1	0.48161	10.26112	36.01448	77.74162
0	0.00002	12.08514	48.34033	108.76566
+1	0.48161	10.26112	36.01450	77.74167
+2	6.68277	10.15723	62.03687	162.32275

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	13.24142	17.34807	21.45474	25.56140
-1	10.09821	10.97350	11.84878	12.72406
0	10.00000	11.40626	12.81252	14.21878
+1	10.09821	10.97350	11.84878	12.72406
+2	13.24142	17.34807	21.45474	25.56140

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	6.68277	10.16260	62.05859	162.37280
-1	0.48161	10.26778	36.04108	77.80141
0	0.00002	12.08559	48.34204	108.76921
+1	0.48161	10.26779	36.04108	77.80147
+2	6.68277	10.16260	62.05859	162.37305

RESULTS FOR K = 6

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	-2	-1	0	+1	+2
-2	0.00000	0.43605	0.55500	0.00895	0.00000
-1	0.00000	0.08603	0.90501	0.00895	0.00000
0	0.00000	0.07837	0.84326	0.07837	0.00000
+1	0.00000	0.00895	0.90501	0.08603	0.00000
+2	0.00000	0.00895	0.55500	0.43605	0.00000

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF J TRIALS GIVEN
THE EXPERIMENT STARTS AT LEVEL I

	2	3	4	5
-2	0.42709	0.56395	0.00895	0.00000
-1	0.99105	0.00895	0.00000	0.0
0	1.00000	0.00000	0.0	0.0
+1	0.99105	0.00895	0.00000	0.0
+2	0.42709	0.56395	0.00895	0.00000

TYPE LOSS FUNCTION: LINEAR

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	15.49113	19.94110	24.39110	28.84108
-1	12.05370	13.00355	13.95340	14.90325
0	11.99999	13.56738	15.13476	16.70213
+1	12.05370	13.00355	13.95340	14.90325
+2	15.49113	19.94110	24.39110	28.84108

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	9.40359	6.25317	52.49707	148.13672
-1	0.31961	9.88820	36.64963	80.60382
0	0.00009	13.21725	52.86861	118.95435
+1	0.31961	9.88820	36.64967	80.60376
+2	9.40359	6.25317	52.49707	148.13672

TYPE_LOSS_FUNCTION:__QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	15,49113	19,94112	24,39111	28,84111
-1	12,05370	13,00356	13,95342	14,90328
0	11,99999	13,56728	15,13476	16,70213
+1	12,05370	13,00356	13,95342	14,90329
+2	15,49113	19,94112	24,39111	28,84111

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR
ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS
VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I

	0	10	20	30
-2	9,40359	6,25366	52,49902	148,14087
-1	0,31961	9,88882	36,65215	80,60945
0	0,00009	13,21724	52,86861	118,95459
+1	0,31961	9,88882	36,65218	80,60939
+2	9,40359	6,25366	52,49902	148,14087

RESULTS FOR OLD PROCEDURE

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN START AT LEVEL I

	-3	-2	-1	0	+1	+2	+3
-3	0.00000	0.01112	0.33349	0.56176	0.09356	0.00007	0.00000
-2	0.00000	0.00083	0.34378	0.56176	0.09356	0.00007	0.00000
-1	0.00000	0.00071	0.16979	0.73488	0.09455	0.00007	0.00000
0	0.00000	0.00009	0.17179	0.65625	0.17179	0.00009	0.00000
+1	0.00000	0.00007	0.09455	0.73488	0.16979	0.00071	0.00000
+2	0.00000	0.00007	0.09356	0.56176	0.34378	0.00083	0.00000
+3	0.00000	0.00007	0.09356	0.56176	0.33349	0.01112	0.00000

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF FIRING J ROUNDS GIVEN START AT LEVEL I

	6	9	12	15	18	21
-3	0.01043	0.24056	0.65538	0.09356	0.00007	0.00000
-2	0.25099	0.65538	0.09356	0.00007	0.00000	0.0
-1	0.90539	0.09455	0.00007	0.00000	0.0	0.0
0	0.99983	0.00017	0.00000	0.0	0.0	0.0
+1	0.90539	0.09455	0.00007	0.00000	0.0	0.0
+2	0.25099	0.65538	0.09356	0.00007	0.00000	0.0
+3	0.01043	0.24056	0.65538	0.09356	0.00007	0.00000

TYPE_LOSS_FUNCTION:__QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	11.49679	16.21487	20.93297	25.65106	30.36916	35.08723
-2	8.52809	12.93752	17.34695	21.75639	26.16582	30.57523
-1	6.28403	8.95852	11.63301	14.30749	16.98198	19.65646
0	6.00052	9.44324	12.88597	16.32869	19.77142	23.21413
+1	6.28403	8.95852	11.63301	14.30750	16.98198	19.65646
+2	8.52809	12.93752	17.34695	21.75639	26.16580	30.57523
+3	11.49679	16.21487	20.93297	25.65106	30.36914	35.08723

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	3.13193	32.30640	138.17480	320.73633	579.99219	915.94336
-2	2.88074	23.20892	94.99976	218.25220	392.96899	619.14941
-1	0.77266	25.48325	91.24463	198.05663	345.91919	534.83325
0	0.00159	22.82376	91.21396	205.17212	364.69727	569.79224
+1	0.77266	25.48325	91.24464	198.05658	345.91919	534.83350
+2	2.88074	23.20894	94.99951	218.25220	392.96973	619.14941
+3	3.13193	32.30640	138.17505	320.73633	579.99316	915.94360

TYPE LOSS FUNCTION: ASYMMETRIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	11.49679	21.94656	32.39632	42.84610	53.29587	63.74564
-2	8.52809	18.87497	29.22183	39.56873	49.91562	60.26250
-1	6.28403	11.40622	16.52840	21.65059	26.77275	31.89494
0	6.00052	11.15764	16.31476	21.47188	26.62898	31.78610
+1	6.28403	9.12308	11.96213	14.80118	17.64023	20.47928
+2	8.52809	11.33752	14.14695	16.95638	19.76579	22.57523
+3	11.49679	14.30622	17.11565	19.92506	22.73450	25.54393

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	3.13193	174.04272	762.44556	1768.33545	3191.70679	5032.57422
-2	2.88074	171.09985	746.68555	1729.63452	3119.93896	4917.61328
-1	0.77266	125.57651	505.80713	1141.46533	2032.55200	3179.06421
0	0.00159	128.16971	512.64282	1153.42114	2050.50562	3203.89380
+1	0.77266	93.34770	340.19727	741.32324	1296.72461	2006.40137
+2	2.88074	98.82063	347.59204	749.19653	1303.63306	2010.90186
+3	3.13193	99.24768	348.19507	749.97559	1304.58765	2012.03223

RESULTS FOR BIO-ASSAY PROCEDURE

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN START AT LEVEL I

	-3	-2	-1	0	+1	+2	+3
-3	0.00000	0.00039	0.09335	0.73663	0.16922	0.00041	0.00000
-2	0.00000	0.00080	0.10116	0.72841	0.16922	0.00041	0.00000
-1	0.00000	0.00090	0.19265	0.63621	0.16982	0.00042	0.00000
0	0.00000	0.00046	0.18637	0.62635	0.18637	0.00046	0.00000
+1	0.00000	0.00042	0.16982	0.63621	0.19265	0.00090	0.00000
+2	0.00000	0.00041	0.16922	0.72841	0.10116	0.00080	0.00000
+3	0.00000	0.00041	0.16922	0.73663	0.09335	0.00039	0.00000

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF FIRING J POUNDS GIVEN START AT LEVEL I

	6	7	8	9	10	11
-3	0.00349	0.08835	0.45408	0.41382	0.04012	0.00014
-2	0.09184	0.45408	0.41382	0.04012	0.00014	0.0
-1	0.54402	0.41558	0.04026	0.00014	0.0	0.0
0	0.91134	0.08835	0.00031	0.0	0.0	0.0
+1	0.54402	0.41558	0.04026	0.00014	0.0	0.0
+2	0.09184	0.45408	0.41382	0.04012	0.00014	0.0
+3	0.00349	0.08835	0.45408	0.41382	0.04012	0.00014

TYPE LOSS FUNCTION: QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	8.39915	11.05695	13.71476	16.37256	19.03036	21.68817
-2	7.40264	10.15492	12.90720	15.65948	18.41176	21.16403
-1	6.49652	10.17395	13.85139	17.52881	21.20625	24.88367
0	6.08897	9.85291	13.61685	17.38078	21.14473	24.90866
+1	6.49652	10.17395	13.85139	17.52881	21.20625	24.88367
+2	7.40264	10.15492	12.90720	15.65948	18.41176	21.16403
+3	8.39915	11.05695	13.71476	16.37256	19.03036	21.68817

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	0.51857	21.44533	83.33028	186.17334	329.97363	514.73340
-2	0.50526	22.49702	87.29736	194.90556	345.32275	538.54883
-1	0.33138	25.41481	100.16231	224.57422	398.64990	622.39136
0	0.08167	25.00333	99.06209	222.25830	394.59204	616.06348
+1	0.33138	25.41481	100.16234	224.57422	398.64990	622.39160
+2	0.50526	22.49702	87.29736	194.90553	345.32275	538.54907
+3	0.51857	21.44531	83.33028	186.17334	329.97363	514.73340

TYPE LOSS FUNCTION: ASYMMETRIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	8.39915	11.21509	14.03104	16.84698	19.66293	22.47888
-2	7.40264	10.46924	13.53584	16.60243	19.66904	22.73564
-1	6.49652	12.31204	18.12756	23.94307	29.75859	35.57411
0	6.08897	11.69833	17.30769	22.91702	28.52640	34.13577
+1	6.49652	11.60788	16.71924	21.83060	26.94194	32.05331
+2	7.40264	12.49579	17.58893	22.68208	27.77522	32.86838
+3	8.39915	13.49230	18.58545	23.67859	28.77173	33.86488

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	0.51857	71.75021	296.39014	674.43994	1205.88819	1890.76538
-2	0.50526	78.44199	322.20532	731.79614	1307.21387	2048.45776
-1	0.33138	136.18564	554.05200	1253.93091	2235.82153	3499.71704
0	0.08167	137.55116	549.01807	1234.48389	2193.94580	3427.40137
+1	0.33138	133.59451	521.62036	1164.40942	2061.96240	3214.27271
+2	0.50526	134.34334	522.22217	1164.14209	2060.10278	3210.10352
+3	0.51857	134.39226	522.30640	1164.26221	2060.25854	3210.29468

RESULTS FOR CURRENT PROCEDURE

PROCEDURE TRANSITION MATRIX

I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J GIVEN START AT LEVEL I

	-3	-2	-1	0	+1	+2	+3
-3	0.00000	0.00085	0.24441	0.51735	0.23662	0.00077	0.00000
-2	0.00000	0.00391	0.18747	0.61965	0.18681	0.00216	0.00000
-1	0.00000	0.00085	0.24441	0.51735	0.23662	0.00077	0.00000
0	0.00000	0.00217	0.18747	0.62073	0.18747	0.00217	0.00000
+1	0.00000	0.00077	0.23662	0.51735	0.24441	0.00085	0.00000
+2	0.00000	0.00216	0.18681	0.61965	0.18747	0.00391	0.00000
+3	0.00000	0.00077	0.23662	0.51735	0.24441	0.00085	0.00000

PROCEDURE ITERATION MATRIX

I-TH ROW IS PROBABILITY OF FIRING J ROUNDS GIVEN START AT LEVEL I

	7	8	9
-3	0.08866	0.83054	0.08080
-2	0.50174	0.49652	0.00174
-1	0.91920	0.08080	0.0
0	0.99651	0.00349	0.0
+1	0.91920	0.08080	0.0
+2	0.50174	0.49652	0.00174
+3	0.08866	0.83054	0.08080

TYPE_LOSS_FUNCTION:__QUADRATIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	7.99214	12.86734	17.74252	22.61774	27.49290	32.36812
-2	7.49999	11.48564	15.47129	19.45692	23.44258	27.42821
-1	7.08080	11.95600	16.83119	21.70639	26.58157	31.45679
0	7.00349	10.92618	14.84888	18.77156	22.69426	26.61693
+1	7.08080	11.95600	16.83119	21.70639	26.58159	31.45679
+2	7.49999	11.48564	15.47129	19.45692	23.44258	27.42821
+3	7.99214	12.86734	17.74252	22.61774	27.49292	32.36812

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	0.16942	27.01624	107.72729	242.30078	430.74146	673.04272
-2	0.25351	31.57288	125.40590	281.75269	500.61304	781.98328
-1	0.07430	27.88077	109.55151	245.08521	434.48462	677.74585
0	0.00351	29.29543	116.66740	262.11987	465.65259	727.26660
+1	0.07430	27.88077	109.55151	245.08521	434.48364	677.74609
+2	0.25351	31.57283	125.40587	281.75269	500.61304	781.98353
+3	0.16942	27.01624	107.72729	242.30078	430.74048	673.04297

TYPE LOSS FUNCTION: ASYMMETRIC

CONDITIONAL EXPECTED LOSS VECTORS

I-TH COMPONENTS ARE EXPECTED LOSSES FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	7.99214	15.35831	22.72447	30.09062	37.45680	44.82297
-2	7.49999	13.28048	19.06096	24.84143	30.62193	36.40242
-1	7.08080	14.44697	21.81313	29.17929	36.54547	43.91164
0	7.00349	12.71418	18.42487	24.13553	29.84624	35.55692
+1	7.08080	14.21040	21.34000	28.46959	35.59918	42.72879
+2	7.49999	13.19075	18.88150	24.57225	30.26300	35.95377
+3	7.99214	15.12174	22.25133	29.38092	36.51051	43.64012

CONDITIONAL VARIANCE VECTORS

I-TH COMPONENTS ARE VARIANCES OF LOSS FOR ABOVE TYPE LOSS FUNCTION
WHEN THE LOSS PARAMETER HAS THE VALUE J GIVEN START AT LEVEL I

	0	10	20	30	40	50
-3	0.16942	162.01280	657.98486	1488.08716	2652.31616	4150.66406
-2	0.25351	136.03835	554.95410	1257.00171	2242.17847	3510.48120
-1	0.07430	165.94798	665.95117	1500.08276	2668.34229	4170.71375
0	0.00351	139.67859	558.50464	1256.48291	2233.61035	3489.88550
+1	0.07430	167.15181	660.96191	1481.50537	2628.78149	4102.78125
+2	0.25351	145.28523	568.72021	1270.55933	2250.80273	3509.44019
+3	0.16942	168.51095	663.58569	1485.39355	2633.93408	4109.19922

```
C
C       THIS PROGRAM COMPUTES THE CONDITIONAL EXPECTED LOSS
C       AND VARIANCE OF THE LOSS FOR PROCEDURES W(δ1,K,0,1)
C       K=1,...,6 FOR A LINEAR OR QUADRATIC LOSS FUNCTION
C       WHERE THE RULE δ1 IS TO STOP WHEN BOTH RESPONSES AND
C       NONRESPONSES HAVE BEEN OBTAINED AT TWO SUCCESSIVE
C       LEVELS.
C
      INTEGER H(5),HSQ(5)
      DIMENSION N(4),NSQ(4),A(5,5),B(5,4),C(5,5),W1(5),W2(5)
     1,W3(5),W4(5),R(5),RP(5,4),E(5),D(5),V(5),VP(5,4)
   95 FORMAT(4I5)
  190 FORMAT('1',////T56,'RESULTS FOR K = ',I2/'+',
     1T56,'_____________________')
  200 FORMAT(///////,T52,'PROCEDURE TRANSITION MATRIX'///
     1T38,'I-TH ROW IS PROBABILITY OF ENDING AT LEVEL J'
     2T38,'GIVEN'//T38,'THE EXPERIMENT STARTS AT LEVEL I'///
     3T47,'-2',8X,'-1',9X,'0',8X,'+1',8X,'+2'///
     4T39,'-2',5F10.5//T39,'-1',5F10.5//T40,'0',5F10.5//T39,
     5'+1',5F10.5//T39,'+2',5F10.5)
  205 FORMAT(///////,T52,'PROCEDURE ITERATION MATRIX'///
     1T43,'I-TH ROW IS PROBABILITY OF J TRIALS GIVEN'//
     2T43,'THE EXPERIMENT STARTS AT LEVEL I'///
     3T52,'2',9X,'3',9X,'4',9X,'5'///
     4T44,'-2',4F10.5//T44,'-1',4F10.5//T45,'0',4F10.5//T44,
     5'+1',4F10.5//T44,'+2',4F10.5)
  220 FORMAT('1',////T50,'TYPE LOSS FUNCTION:  ')
  222 FORMAT('+',T71,'LINEAR'/'+',
     1T50,'_____________________________')
  223 FORMAT('+',T71,'QUADRATIC'/'+',
     1T50,'_____________________________')
  230 FORMAT(///////,T48,'CONDITIONAL EXPECTED LOSS VECTORS'/
     1//T39,'I-TH COMPONENT IS EXPECTED TOTAL LOSS FOR'//
     2T39,'ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS'
     3//T39,'VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I'///
     4T50,'0',10X,'10',10X,'20',10X,'30'///
     5T40,'-2',4F12.5//T40,'-1',4F12.5//T41,'0',4F12.5//T40,
     6'+1',4F12.5//T40,'+2',4F12.5)
  240 FORMAT(///////,T50,'CONDITIONAL VARIANCE VECTORS'///
     1T39,'I-TH COMPONENT IS VARIANCE OF TOTAL LOSS FOR'//
     2T39,'ABOVE TYPE LOSS FUNCTION WHEN LOSS PARAMETER HAS'
     3//T39,'VALUE J GIVEN EXPERIMENT STARTS AT LEVEL I'///
     4T50,'0',10X,'10',10X,'20',10X,'30'///
     5T40,'-2',4F12.5//T40,'-1',4F12.5//T41,'0',4F12.5//T40,
     6'+1',4F12.5//T40,'+2',4F12.5)
C
C       INPUT THE VECTORS N AND NSQ.
C
      READ(5,95)(N(I),I=1,4)
      READ(5,95)(NSQ(I),I=1,4)
C
C       K IS THE NUMBER OF OBSERVATIONS PER TRIAL.
C
      DO 50 K=1,6
C
C       COMPUTE A, B, AND C MATRICES.
C
      CALL PRO(K,A,B,C)
      WRITE(6,190)K
      WRITE(6,200)((A(I,J),J=1,5),I=1,5)
      WRITE(6,205)((B(I,J),J=1,4),I=1,5)
C
C       PERFORM COMPUTATIONS FOR TWO LOSS FUNCTIONS USING
C       FOUR VALUES OF THE LOSS PARAMETER.
C
      DO 50 I=1,2
      DO 25 J=1,4
C
C       THE FOLLOWING SUBROUTINES COMPUTE THE VALUES OF THE
C       COMPONENTS OF R=AH+KBN, THE CONDITIONAL LOSS VECTOR,
C       AND STORE IN A MATRIX FOR OUTPUT LATER.
C
```

```
      CALL LOSS(I,J,H,HSQ)
      CALL PROD1(5,A,H,W1)
      CALL PROD1(4,B,N,W2)
      CALL PROD3(K,W2,W3)
      CALL SUM(W1,W3,R)
      CALL TABLE1(J,R,RP)
C
C     THE FOLLOWING SUBROUTINES COMPUTE THE VALUES OF THE
C     COMPONENTS OF V=C-DIAGONAL OF D, THE CONDITIONAL
C     VARIANCE VECTOR, AND STORE IN A MATRIX FOR OUTPUT
C     LATER.
C
      CALL PROD1(5,A,HSQ,W1)
      CALL PROD1(4,B,NSQ,W2)
      K1=K*K
      CALL PROD3(K1,W2,W3)
      CALL SUM(W1,W3,W4)
      CALL PROD1(5,C,H,W1)
      K1=2*K
      CALL PROD3(K1,W1,W2)
      CALL SUM(W4,W2,E)
      CALL PROD2(E,R,W1)
      K1=-1
      CALL PROD3(K1,W1,D)
      CALL SUM(E,D,V)
      CALL TABLE1(J,V,VP)
      IF(J.EQ.1)WRITE(6,220)
      IF(I.EQ.1.AND.J.EQ.1)WRITE(6,222)
      IF(I.EQ.2.AND.J.EQ.1)WRITE(6,223)
   25 CONTINUE
C
C     OUTPUT THE CONDITIONAL EXPECTED LOSS AND VARIANCE
C     OF THE LOSS.
C
      WRITE(6,230)((RP(L,M),M=1,4),L=1,5)
   50 WRITE(6,240)((VP(L,M),M=1,4),L=1,5)
      STOP
      END


      SUBROUTINE PRO(K,A,B,C)
C
C     THIS SUBROUTINE COMPUTES ELEMENTS OF THE A, B, AND C
C     MATRICES.  E(I,N,J) IS JOINT CONDITIONAL PROBABILITY
C     THAT THE EXPERIMENT ENDS AT LEVEL I AFTER N TRIALS
C     GIVEN IT STARTS AT LEVEL J.
C
      DIMENSION E(5,4,5),A(5,5),B(5,4),C(5,5),G(3)
      BETA1=0.28866
      BETA2=0.5
      ALPHA1=1-BETA1
      P1=(BETA1)**K
      Q1=(ALPHA1)**K
      R1=1-P1-Q1
      P2=(BETA2)**K
      R2=1-2*P2
      DO 5 I=1,5
      DO 5 N=1,4
      DO 5 J=1,5
    5 E(I,N,J)=0
      E(1,1,1)=P1
      E(2,1,1)=R1
      E(2,2,1)=P2+Q1
      E(3,2,1)=R2+Q1
      E(4,2,1)=P2+Q1*(1-P1)
      E(5,4,1)=R2+P1*Q1
      CALL GAMMA(K,-1,BETA1,G)
      E(1,1,2)=P1+G(1)
      E(2,1,2)=E(2,2,1)+G(2)
      E(3,1,2)=E(3,2,1)+G(3)
      E(4,2,2)=E(4,3,1)
      E(5,3,2)=E(5,4,1)
```

```
      CALL GAMMA(K,0,BETA0,G)
      F(1,2,3)=P0*P1
      F(2,1,3)=P0*(1-P1)+G(1)
      F(4,1,3)=G(2)
      F(4,2,3)=P0*(1-P1)+G(3)
      F(5,2,3)=F(1,2,3)
      CALL GAMMA(K,1,ALPHA1,G)
      F(1,3,4)=F(5,3,2)
      F(2,2,4)=F(4,2,2)
      F(3,1,4)=F(1,2,1)+G(1)
      F(4,1,4)=F(5,2,1)+G(2)
      F(5,1,4)=P1+G(3)
      F(1,4,5)=F(1,3,4)
      F(2,2,5)=F(2,2,4)
      F(3,2,5)=F(3,2,1)
      F(4,1,5)=F(2,1,1)
      F(4,2,5)=F(2,2,1)
      F(5,1,5)=F(1,1,1)
      DO 10 I=1,5
      DO 10 J=1,5
      A(I,J)=0
      C(I,J)=0
      DO 10 N=1,4
      A(I,J)=A(I,J)+F(J,N,I)
   10 C(I,J)=C(I,J)+(N+1)*F(J,N,I)
      DO 15 I=1,5
      DO 15 N=1,4
      B(I,N)=0
      DO 15 J=1,5
   15 B(I,N)=B(I,N)+F(J,N,I)
      RETURN
      END


      SUBROUTINE GAMMA(K,Q,ALPHA,P)
C
C     THIS SUBROUTINE COMPUTES A FRACTIONAL PORTION OF
C     PR(M=I|Q=I), PR(M=I-1|Q=I) AND PR(M=I+1|Q=I); I=-1,0,1
C
      DIMENSION P(3)
      IF(K.EQ.1)GO TO 150
      BETA=1.0-ALPHA
      JMAX=K-1
      CALL FACT(K,KF)
      K0=K/2
      K1=K0
      FK=K/2.0-K0
      IF(FK.GT.0)K1=K0+1
      K2=2*K
      M=1
    5 SUMJN=0.0
      DO 100 J=1,JMAX
      KJ=K-J
      CALL FACT(J,JF)
      CALL FACT(KJ,KJF)
      KC=KF/(JF*KJF)
      X1=ALPHA**J
      X2=BETA**KJ
      PJ1=KC*X1*X2
      IF(Q) 10,20,30
C
C     Q=-1, 0, OR +1 IS THE STARTING LEVEL.
C
   10 GO TO (11,12,13), M
C
C     M=1, 2, OR 3 MEANS:  IF 1, GO DOWN A LEVEL;
C                          IF 2, STAY AT THE SAME LEVEL;
C                          IF 3, GO UP A LEVEL.
C
   11 NMIN=K2-K0-J
      NMAX=K2-1-J
      GO TO 40
```

```
   12 NMIN=K1-J
      NMAX=K2-K0-1-J
      GO TO 40
   13 NMIN=1-J
      NMAX=K1-1-J
      GO TO 40
   20 GO TO (21,22,23), M
   21 NMIN=K2-K0-J
      NMAX=K2-1-J
      GO TO 40
   22 NMIN=K0+1-J
      NMAX=K2-K0-1-J
      GO TO 40
   23 NMIN=1-J
      NMAX=K0-J
      GO TO 40
   30 GO TO (31,32,33), M
   31 NMIN=K2-K1+1-J
      NMAX=K2-1-J
      GO TO 40
   32 NMIN=K0+1-J
      NMAX=K2-K1-J
      GO TO 40
   33 NMIN=1-J
      NMAX=K0-J
   40 IF(NMIN.LT.0.AND.NMAX.LT.0)GO TO 41
      IF(NMIN.GT.K.AND.NMAX.GT.K)GO TO 41
      IF(NMIN.GT.NMAX)GO TO 41
      IF(NMIN.LT.0.AND.NMAX.GE.0)NMIN=0
      IF(NMIN.LE.K.AND.NMAX.GT.K)NMAX=K
      IF(NMIN.EQ.0.AND.NMAX.EQ.0)GO TO 42
      IF(NMIN.EQ.K.AND.NMAX.EQ.K)GO TO 43
      IF(NMIN.EQ.0.AND.NMAX.GT.0)GO TO 44
      GO TO 48
   41 PJ2=0.0
      GO TO 100
   42 PJ2=BETA**K
      GO TO 100
   43 PJ2=ALPHA**K
      GO TO 100
   44 PJ2=BETA**K
      NMIN=1
      GO TO 49
   48 PJ2=0.0
   49 DO 50 N=NMIN,NMAX
      KN=K-N
      CALL FACT(N,NF)
      CALL FACT(KN,KNF)
      NC=KF/(NF*KNF)
      Y1=ALPHA**N
      Y2=BETA**KN
   50 PJ2=PJ2+NC*Y1*Y2
  100 SUMJN=SUMJN+PJ1*PJ2
      P(M)=SUMJN
      IF(M.EQ.3)RETURN
      M=M+1
      GO TO 6
  150 DO 155 I=1,3
  155 P(I)=0.0
      RETURN
      END


      SUBROUTINE FACT(N,NF)
C
C     THIS SUBROUTINE COMPUTES N FACTORIAL FOR INTEGER
C     N=0,1,...,10.
C
      I=N
      NF=1
      IF(N.LE.0)GO TO 15
   10 IF(N.GT.0)GO TO 20
```

```
   15 N=I
      RETURN
   20 NF=NFAM
      N=N-1
      GO TO 1
      END


      SUBROUTINE LOSS(N,J,H,HSQ)
C
C     THIS SUBROUTINE COMPUTES LOSS VECTORS H AND HSQ FOR
C     N TYPES OF LOSS FUNCTIONS:
C        1 - LINEAR
C        2 - QUADRATIC
C     THE LOSS PARAMETER ETA HAS VALUES 5, 10, 20, AND 30.
C
      INTEGER ETA,H(5),HSQ(5)
      ETA=(J-1)*10
      IF(J.EQ.1)GO TO 10
      GO TO (1,2), N
    1 DO 15 I=1,2
   15 H(I+3)=I*ETA
      DO 12 I=1,2
   12 H(I)=H(4-I)
      H(3)=0
      GO TO 40
    2 DO 20 I=1,2
   20 H(I+3)=I*I*ETA
      GO TO 11
   10 DO 31 I=1,5
   31 H(I)=ETA
   11 DO 41 I=1,5
   41 HSQ(I)=H(I)*H(I)
      RETURN
      END


      SUBROUTINE PROD1(N,X,A,B)
C
C     THIS SUBROUTINE POST-MULTIPLIES A MATRIX BY A VECTOR.
C
      INTEGER A(5)
      DIMENSION X(5,5),B(5)
      DO 15 I=1,5
      B(I)=0
      DO 10 J=1,N
   10 B(I)=B(I)+X(I,J)*A(J)
      RETURN
      END


      SUBROUTINE PROD2(A,B,X)
C
C     THIS SUBROUTINE COMPUTES THE DIAGONAL OF THE D MATRIX.
C
      DIMENSION A(5),B(5),X(5)
      DO 10 I=1,5
   10 X(I)=A(I)*B(I)
      RETURN
      END


      SUBROUTINE PROD3(K,A,B)
C
C     THIS SUBROUTINE MULTIPLIES A VECTOR BY A CONSTANT.
C
      DIMENSION A(5),B(5)
      DO 10 I=1,5
   10 B(I)=K*A(I)
      RETURN
      END
```

```
      SUBROUTINE SUM(A,B,C)
C
C     THIS SUBROUTINE SUMS TWO VECTORS.
C
      DIMENSION A(5),B(5),C(5)
      DO 10 I=1,5
   10 C(I)=A(I)+B(I)
      RETURN
      END

      SUBROUTINE TABLE1(N,A,X)
C
C     THIS SUBROUTINE STORES A VECTOR IN THE N-TH COLUMN
C     OF A MATRIX.
C
      DIMENSION A(5),X(5,4)
      DO 10 I=1,5
   10 X(I,N)=A(I)
      RETURN
      END
C
C     DATA CARDS
C
    2     3     4     5
    4     9    16    25
```

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13. ABSTRACT

A class of sequential procedures for estimating the mean of a normal distribution having known variance from quantal response data is discussed. This class includes as special members the up-and-down method and other procedures commonly used in biological assay. A method of evaluating alternative procedures belonging to a given subset of the class is presented. This method is essentially an application of Wald's decision theory. A loss plus cost objective function is used and the efficiency of a particular procedure is determined by its ability to satisfy one of the four criteria considered. Criteria are discussed for use with both the expected value and variance of the total loss, which may be determined from matrix equations that are derived. Two applications are given. The first is an application to procedures commonly used in biological assay. In the second, an application to the elevation procedure of the precision registration technique used by U.S. Army and Marine Corps field artillery units, it is seen that under certain conditions, Dixon's modified up-and-down method strictly dominates the elevation procedure currently in use.

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UP-AND-DOWN METHOD						
DECISION THEORY						
FIELD ARTILLERY PRECISION REGISTRATION						
RANDOM WALK DESIGN						

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c.1 Evaluation of a class
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procedures.